Self-similarity of Vorticity Dynamics in Decaying Two-dimensional Turbulence

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1 Introduction

1.1 Background

- Batchelor(1969)
  - Assumed that the energy $\mathcal{E}$ is the unique invariants.
  - Proposed the self-similar energy spectrum and the enstrophy decay law,

$$E(k) = \mathcal{E}^{3/2} t F(k\mathcal{E}^{1/2}), \quad \mathcal{Q} \sim t^{-2}, \tag{1}$$

  for high Re limit.
  - However, numerical experiments contradict this prediction.
- Carnevale *et al.* (1991)
  - Proposed the vortex scaling theory

$$\mathcal{E} \sim t^0 \quad \text{and} \quad \mathcal{Q} \sim t^{-\xi/2}, \quad \text{numerically } \xi = 0.7 - 0.75 \tag{2}$$

  - Supported by many DNSs using hyperviscosity.
- Bartello & Warn (1996)
  - The existence of coherent vortices $\Longrightarrow$ the existence of the second invariant, $\omega_{\rm m}$, (the vorticity of the most intense vortex) $\Longrightarrow$ spoil the Batchelor's assumption.
  - The one-point vorticity PDF is self-similar in the core, not in the wings corresponding to the coherent vortices.
    - ∗ This corresponds to a phase transition of the vorticity moments:

$$\langle|\omega|^q\rangle \sim \begin{cases} c_q t^{-q}, & (-1 < q < q_c), \\ c(q-q_c)^{-1}\omega_{\rm m}^{q-q_c} t^{-q_c}, & (q > q_c), \end{cases} \tag{3}$$

$$(q_c \approx 0.4) \tag{4}$$

    - ∗ For finite Re case, (finite Re vortex scaling theory)

$$\omega_{\rm m} \sim t^{-\chi}. \tag{5}$$

- Chasnov (1997)
  - Numerically found some self-similar solutions of the form

$$E(k) = \mathcal{E} l F(kl), \quad l \equiv \sqrt{\mathcal{E}/\mathcal{Q}} \tag{6}$$

  - For $Re = 15.73$,

$$\mathcal{E} \sim t^{-1} \quad \text{and} \quad \mathcal{Q} \sim t^{-2}, \quad \text{(the critical } Re \text{ decay law)} \tag{7}$$

  - For high Re,

$$\mathcal{E} \sim t^{0} \quad \text{and} \quad \mathcal{Q} \sim t^{-1}, \quad \text{(the high } Re \text{ decay law)} \tag{8}$$

  - DNSs using the normal viscosity (Bartello & Warn 1996, Das *et al.* 2001) also obtained same results with (8).
- Eqn. (8) is at variance with the prediction of vortex scaling theory (2).
- Connection between self-similarity and vortex scaling theory has been left as an unsolved problem.

1.2 Purpose of this work

1. Propose a similarity theory for the decaying 2-D NS turbulence, including the viscous range. $\Longrightarrow$ Proceedings
2. Discuss the failure of Batchelor's similarity hypothesis within our framework. $\Longrightarrow$ Proceedings
3. Discuss connection between self-similarity and vortex scaling theory.

2 Theory

- Propose self-similarity of the energy spectrum and energy transfer function, through the entire range (including the viscous range)
  - The self-similar form of $E(k)$

$$E(k) = c\Lambda^{\sigma} t^{\delta} G(x), \quad x \equiv k\Lambda. \tag{9}$$

  From (9) follows

$$\mathcal{E} \sim t^{-\theta}, \quad \mathcal{Q} \sim t^{-\theta-1/p}, \quad \theta = \frac{(1-\sigma)}{2p} - \delta. \tag{10}$$

    - ∗ σ and δ; constants.
    - ∗ c; a constant with the dimension of $(\text{length})^{3-\sigma}/(\text{time})^{\delta+2}$. $\Longrightarrow$ related to $\omega_{\rm m}$
    - ∗ G; a positive definite function of universal form.
    - ∗ Λ; a length scale (any length scale).
    - ∗ p; degree of hyperviscosity
  - The above decay law include the critical Re decay law and the high Re decay law. (Generalization of Chasnov & Herring (1998))
- Our self-similarity theory based on the inviscid equations predicts an upscale energy flux for all wave numbers, in violation of Fjørtoft's theorem.

3 Numerical results

- Our theory predicts $\mathcal{Q} \sim t^{-1/p}$ in the high Re.
- Test our similarity theory.

3.1 Simulation conditions

- The initial energy spectrum

$$E(k) \propto k^7 \exp\left[-\frac{7}{2}\left(\frac{k}{k_p}\right)^2\right], \quad k_p = 40. \tag{11}$$

- Ensemble average over 8 realizations.

	run p1	run p2	run p3	run p4
p	1	2	3	4
N^2	1024^2	512^2	512^2	512^2
$k_{\rm T}$	341	170	170	170
ν_p	9.12949×10^{-5}	3.5×10^{-9}	2.42214×10^{-13}	8.38111×10^{-18}
$\mathcal{E}(0)$	0.5	0.5	0.5	0.5
$\mathcal{Q}(0)$	915.37	915.37	915.37	915.37
$R_l^{(p)}$	256	3648	28790	454470

Table 1: Simulation parameters. p is the degree of hyperviscosity, N^2 the grid points in the simulation, $k_{\rm T}$ the truncation wavenumber, ν_p the viscosity coefficient, $\mathcal{E}(0)$ the initial energy, $\mathcal{Q}(0)$ the initial enstrophy, and $R_l^{(p)}$ the Reynolds number defined by $R_l^{(p)} = \frac{\sqrt{2\mathcal{E}}l^{2p-1}}{\nu_p}$.

3.2 Results

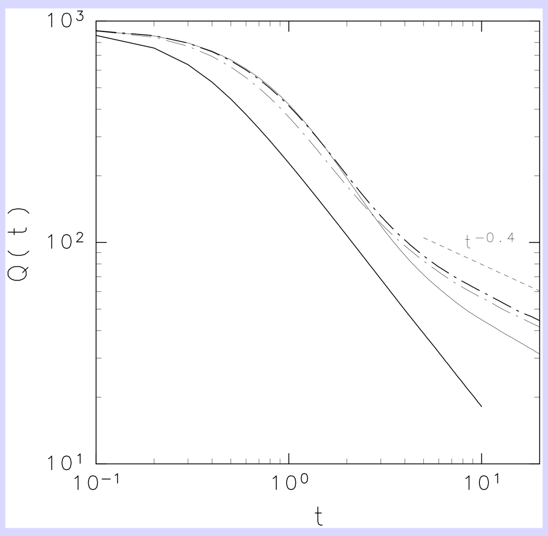

Figure 1: Time evolution of enstrophy. The thick solid line, run p1; the thin solid line, for run p2; the thin dash-dot line, run p3; the thick dash-dot line, run p4. The dotted line indicates the slope of -0.4.

	energy	enstrophy
run p1	$(1.03 \pm 0.01) \times 10^{-1}$	1.10 ± 0.02
run p2	$(6.94 \pm 0.21) \times 10^{-3}$	$(5.13 \pm 0.30) \times 10^{-1}$
run p3	$(3.27 \pm 0.36) \times 10^{-3}$	$(4.39 \pm 0.44) \times 10^{-1}$
run p4	$(2.37 \pm 0.38) \times 10^{-3}$	$(4.31 \pm 0.50) \times 10^{-1}$

Table 2: Decay exponents of energy and enstrophy. The values are estimated by the averages of $t\mathcal{E}/\epsilon$ and $t\mathcal{Q}/\eta$ over the ranges $1.5 \le t \le 10$ for run p1 and $10 \le t \le 20$ for the others.

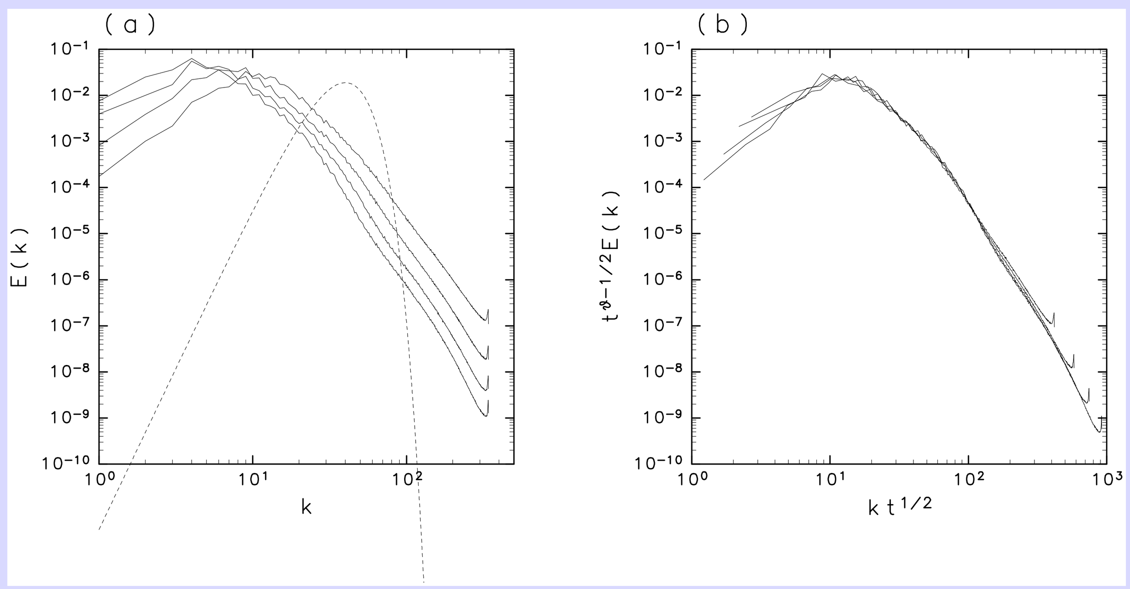

Figure 2: (a) Energy spectra at $t = 1.5$, 2.9, 4.8 and 7.3 for run p1. These times correspond to $\tau \equiv \int_0^t \{2\mathcal{Q}(t')\}^{1/2}\,{\rm d}t' = 41$, 60, 80 and 100. The dotted line indicates the initial energy spectrum. (b) Same as (a) but written in terms of similarity variables. A value of $\theta = 0.1$ is used in drawing this figure.

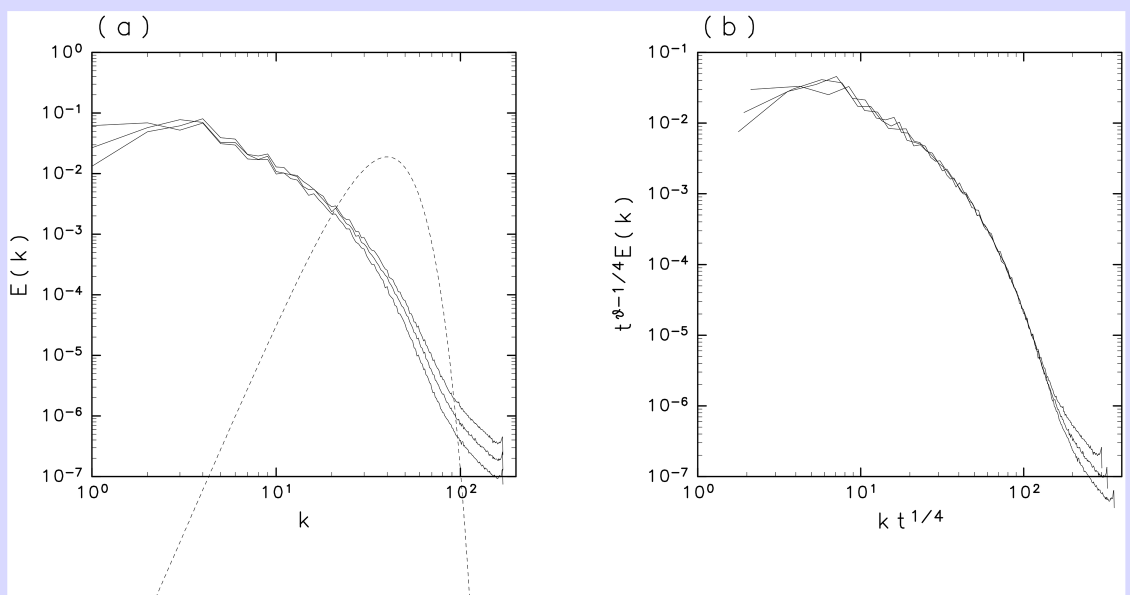

Figure 3: (a) Energy spectra at $t = 10$, 13.6 and 20 for run p2. These times correspond to $\tau = 157$, 186 and 242. The dotted line indicates the initial energy spectrum. (b) Same as (a) but written in terms of similarity variables. A value of $\theta = 0.0069$ is used in drawing this figure.

4 Connection between self-similarity and vortex scaling theory

- The numerical simulations show that our similarity theory applies for $p = 1$ & 2, but not for $p > 2$.
- To understand the failure of our self-similarity theory.
- Equating (3) with $q = 2$ with (10),

$$\chi = \frac{\theta + \frac{1}{p} - q_c}{2 - q_c} > 0, \quad \Longrightarrow \quad \theta + \frac{1}{p} > q_c \tag{12}$$

- As Re increase, $\theta \to 0$. $\Longrightarrow$ $\frac{1}{p} > q_c \approx 0.4$ $\Longrightarrow$ Self-similar evolution of $E(k)$ and the finite Re vortex scaling theory can coexist only for $p = 1$ or $p = 2$, not for $p > 2$.

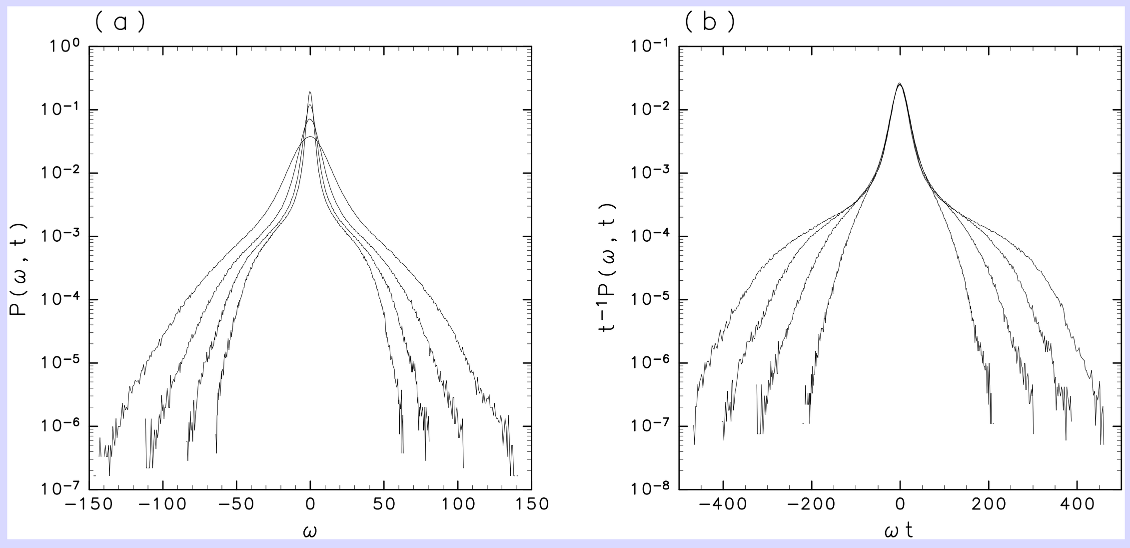

Figure 4: (a) One-point vorticity PDF for run p1 at $t = 1.5$, 2.9, 4.8, 7.3. (b) Same as (a) but written in terms of similarity variables. The PDF gradually narrows, while the scaled PDF broadens with time.

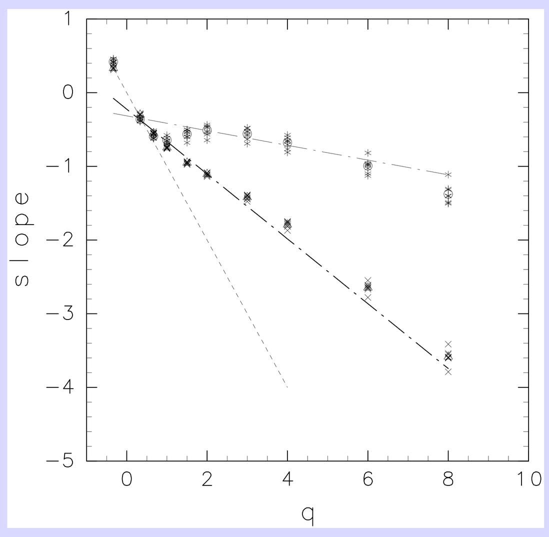

Figure 5: Slopes of the vorticity moments over the ranges $1.5 \le t \le 10$ for run p1 and $10 \le t \le 20$ for run p2. The crosses indicate the slopes for each realization for run p1 and the triangles for the average over their eight realizations.The asterisks indicate the slopes for each realization for run p2 and the open circles for the average over their eight realizations. The dotted line corresponds to prediction from Batchelor's similarity hypothesis, and the thick and thin dash-dot lines correspond to those from the finite Reynolds number modification of vortex scaling theory for run p1 and run p2, respectively.

5 Summary

- We propose self-similarity of the energy spectrum and energy transfer function, through the entire range (including the viscous range)
- Our self-similarity theory predicts $\mathcal{Q} \sim t^{-1/p}$ in the high Re limit.
- The self-similarity involves a dimensional prefactor, suggesting a hidden variable, which is the vortex strength as suggested by Bartello & Warn (1996).
- Numerical simulations show that our theory holds for $p = 1$ and $p = 2$, but not for $p > 2$.
- We can reconcile self-similarity with vortex scaling theory for $p = 1$ and $p = 2$, but not for $p > 2$.
- The implication is that viscosity is never ignorable in decaying 2-D NS turbulence, even in the high Re limit – *i.e.* the inviscid limit is singular.
- Furthermore high Re for usual viscosity cannot be mimicked with hyperviscosity.

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