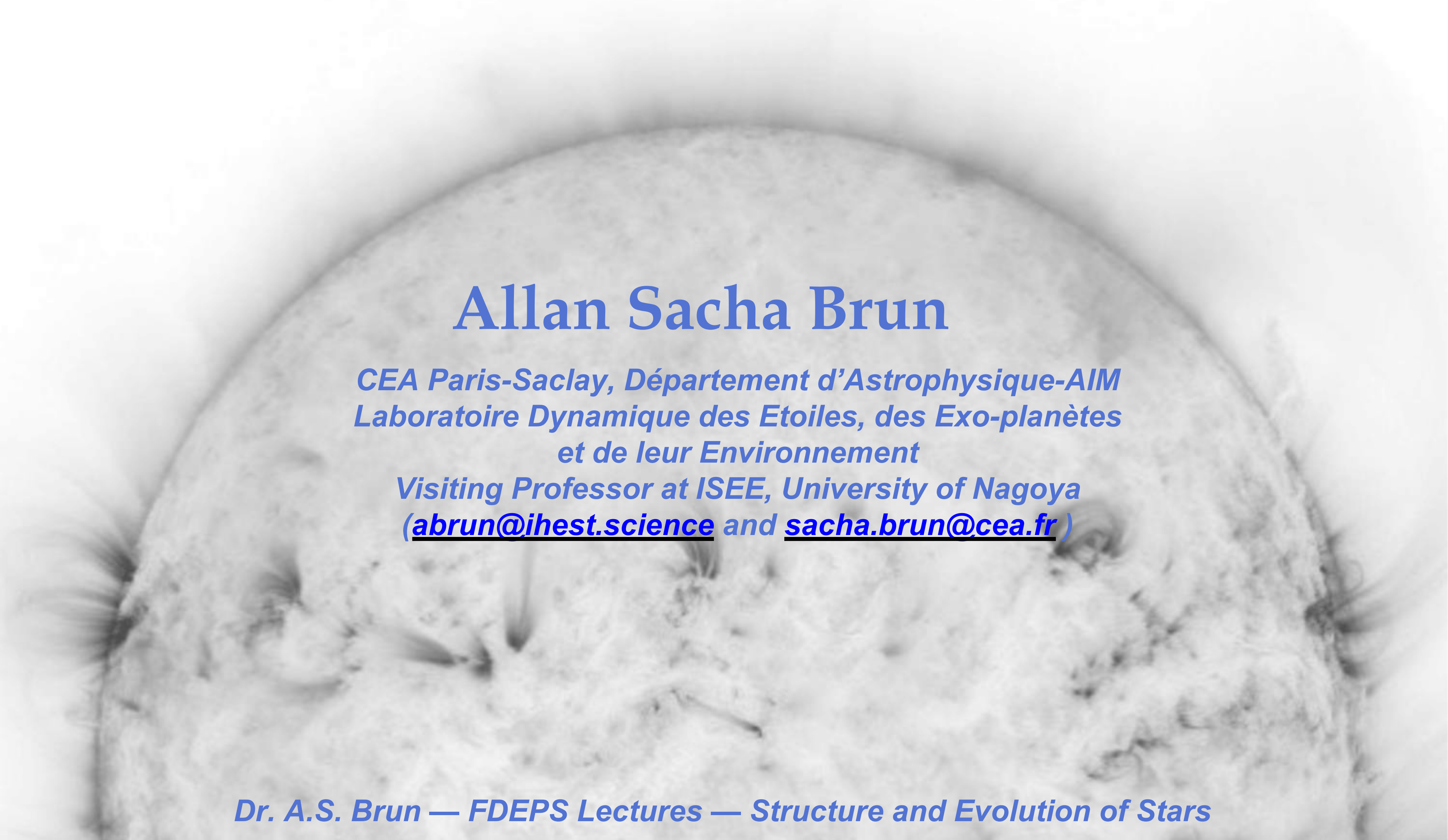
Structure and Evolution of Stars
Turbulent convection and dynamo effect



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Lecture plan

I. Energy transport in stars

II. Stellar fluid dynamics
- From particle description to fluid description in plasmas
- Turbulence: basic concepts
- Convection simulation: some exemples

III. Dynamo effect in turbulent convective zones of stars
- The Sun's magnetic field
- The dynamo effect: fundamental ingredients
- The dynamo effect: kinematics vs. Dynamics
- Stellar magnetism and dynamo

Le transport d'énergie

We saw in the first session that there is a sharp contrast in temperature between the core of a star (~ MK) and its surface (~ a few kK).

A transport of energy must hence starts inside the star

Three modes of energy transport can occur:

radiation
convection
conduction

Radiative transport

Radiative transport is directly related to the opacity of the plasma, characterized by an absorption coefficient κ (cm^2/g).

The photon mean free path in the Sun can be written as

$$\lambda_{\rm mfp} = \frac{1}{\kappa\rho}$$ ~ 1 cm in the Sun

$\tau_{exit} \sim R^2/(\lambda_{\rm mfp}\, c)$

The Sun's photon exit time is then ~ 6000 years! In practice, it is longer (few 10^5 years) because a random "Brownian motion" must be applied.

Since $\frac{\lambda_{\rm mfp}}{R_\odot} \ll 1$, we can then treat the transport as a diffusion

But be careful, at the surface the density drops dramatically and this is no longer true...

Radiative transport

Fick's Law

$$\mathbf{F} = -D\nabla U$$

$$\frac{L(r)}{4\pi r^2}$$

$$\frac{\lambda_{\rm mfp} v}{3} = \frac{c}{3\kappa\rho}$$

$$\frac{4\sigma}{3c}$$

$$U = aT^4$$

Radiation energy density

We then get

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\kappa\rho}{4acT^3}\frac{L(r)}{4\pi r^2}$$

Convective transport: Schwarzschild criterion

Let us assume, then, that energy is transported by radiation within the star.

The temperature gradient is given by the relation (see first lecture)

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\kappa\rho}{4acT^3}\frac{L(r)}{4\pi r^2}$$

If the latter is smaller than the 'adiabatic' gradient, then convection does not develop. This is Schwarzschild's criterion.

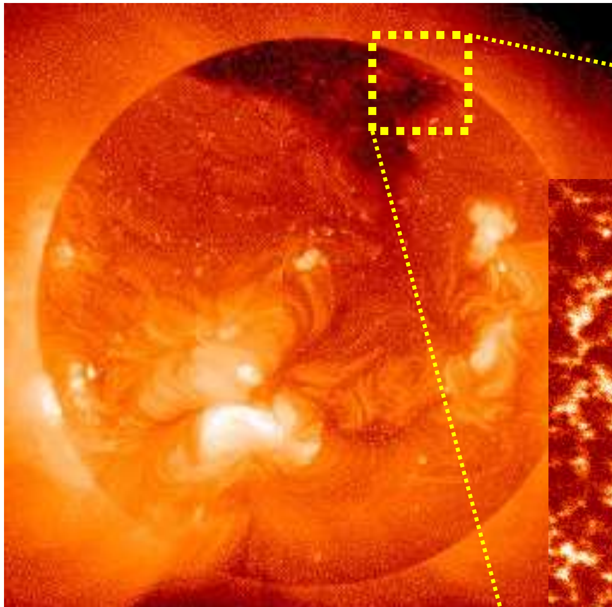
$$\boxed{\nabla_{\mathrm{ad}} > \nabla_{\mathrm{rad}}}$$

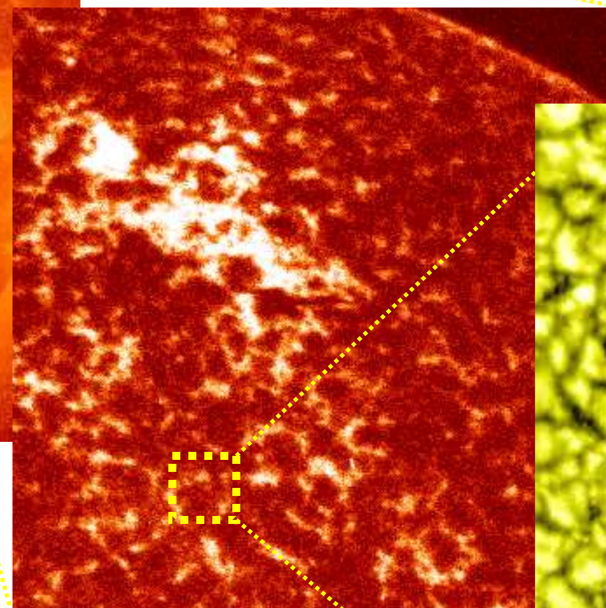
$$\nabla_{\mathrm{ad}} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_s \qquad \nabla_{\mathrm{rad}} = \frac{3\kappa P}{4acGM(r)T^4}\frac{L(r)}{4\pi}$$

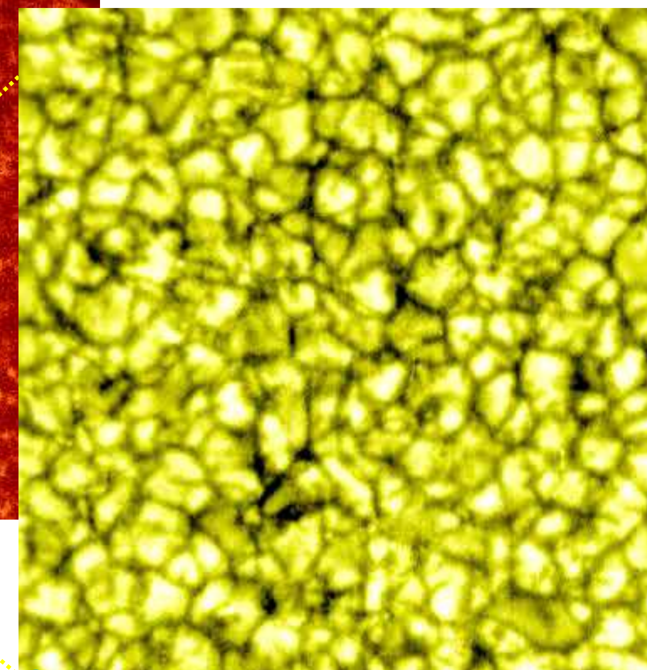
Spatio-Temporal Scales in the Solar Convection Zone

Plasma= hot gas (ionised)
4th state of matter

Order in Chaos!







Big structures:

Eruptions,
Coronal holes,
CMEs

Giant cells?

200+ Mm
10-20 d

Supergranulation:

30-50 Mm
20 h

Mesogranulation?

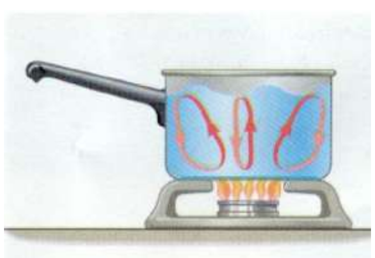
7-10 Mm
2 h

Granulation:

1-2 Mm
5 min

Tiny structures:

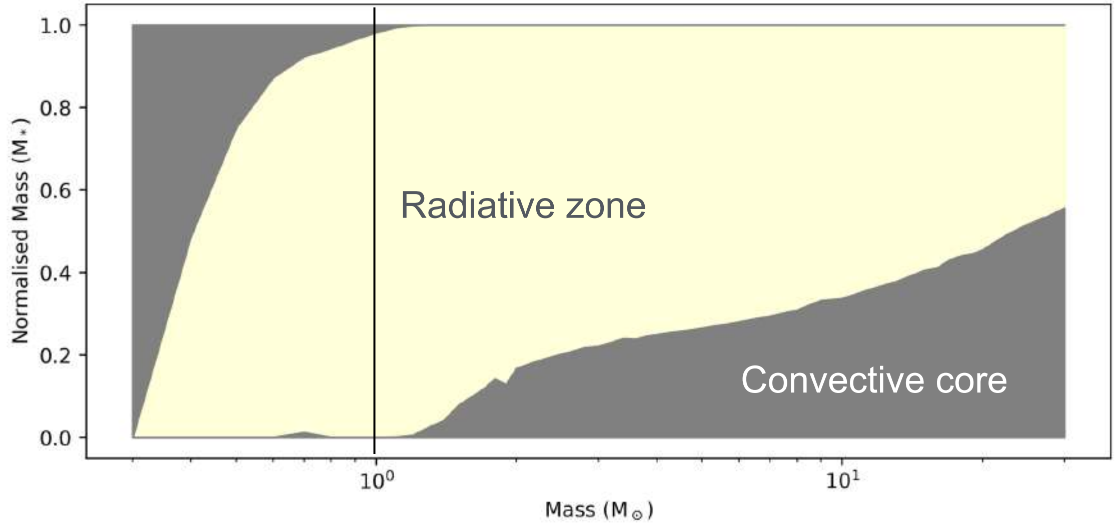
Intergranular lanes,
Bright points
Diffusion



Convective transport: summary for stars

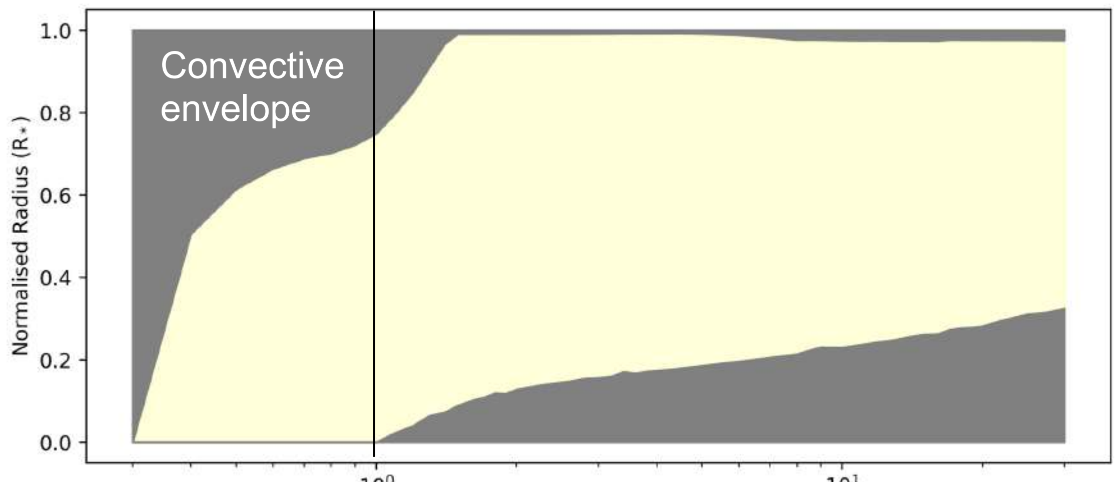
Dwarfs Sun Massives

vs mass



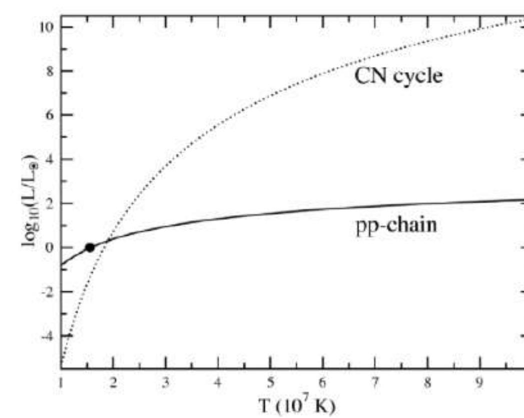
Localization of Convectives zones In stars half way through their main sequence M_*=[0.3-30] M_{sol}

vs radius



Stellar Mass

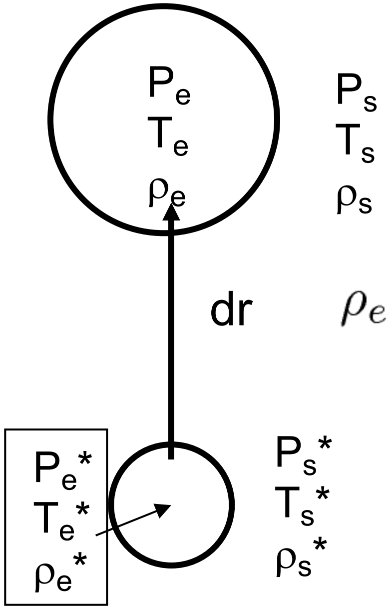
Transition PP->CNO at M*=1.3Msol



PP chains CNO cycle

Amard et al. 2019

Derivation of convection stability criterion



$$\rho_e = \rho_e^* + dr\left(\frac{d\rho}{dr}\right)_e \qquad \rho_s = \rho_s^* + dr\left(\frac{d\rho}{dr}\right)_s$$

$$\rho_e - \rho_s = dr\left[\left(\frac{d\rho}{dr}\right)_e - \left(\frac{d\rho}{dr}\right)_s\right] > 0 \quad \left(\frac{d\rho}{dr}\right)_e - \left(\frac{d\rho}{dr}\right)_s > 0 \ (1)$$

$\rho_e^* = \rho_s^*$

Equation of state (EoS): $\rho=\rho(P,T,\mu)$

$$\frac{d\rho}{\rho} = \alpha\frac{dP}{P} - \delta\frac{dT}{T} + \phi\frac{d\mu}{\mu}$$

Pressure scale height

$$\frac{1}{H_p} = -\frac{d\ln P}{dr}$$

$$\left(\frac{\alpha}{P}\frac{dP}{dr}\right)_e - \left(\frac{\delta}{T}\frac{dT}{dr}\right)_e - \left(\frac{\alpha}{P}\frac{dP}{dr}\right)_s + \left(\frac{\delta}{T}\frac{dT}{dr}\right)_s - \left(\frac{\phi}{\mu}\frac{d\mu}{dr}\right)_s > 0$$

$P_e = P_s$ et $P_e^* = P_s^*$ Composition variation $d\mu$ null for element

Hint:
$dT/dr=(dT/dP).(dP/dr) \Rightarrow (\delta/T \,.\, dT/dr)_s = (\delta P/T \,.\, dT/dP.\, d\ln P/dr)_s = -1/H_p.(\delta\, d\ln T/d\ln P)_s$

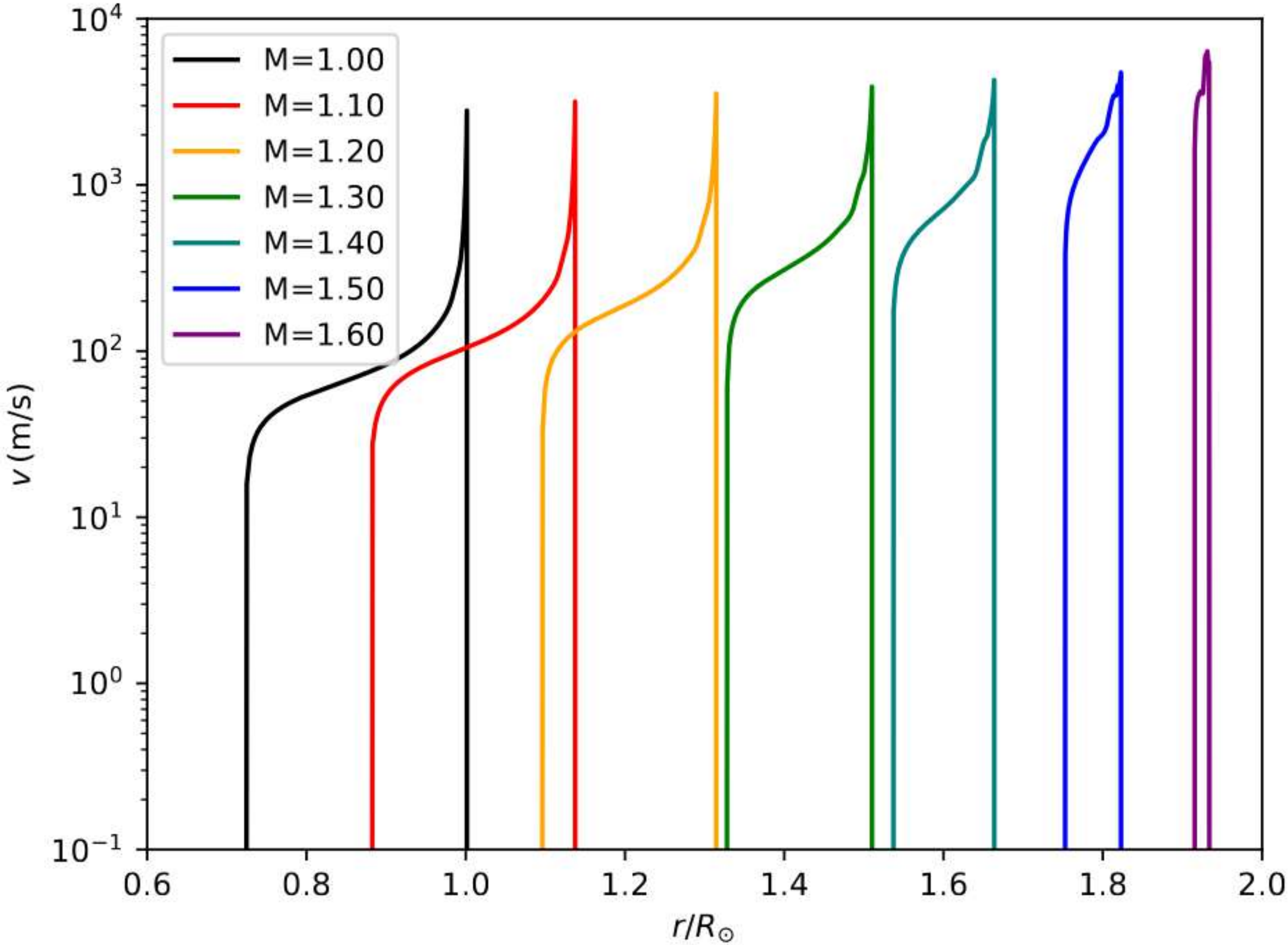
Stable Case

Let's multiply by H_p:

$$\left(\frac{d\ln T}{d\ln P}\right)_s < \left(\frac{d\ln T}{d\ln P}\right)_e + \frac{\phi}{\delta}\left(\frac{d\ln \mu}{d\ln P}\right)_s \text{ or } \nabla < \nabla_e + \frac{\phi}{\delta}\nabla_\mu$$

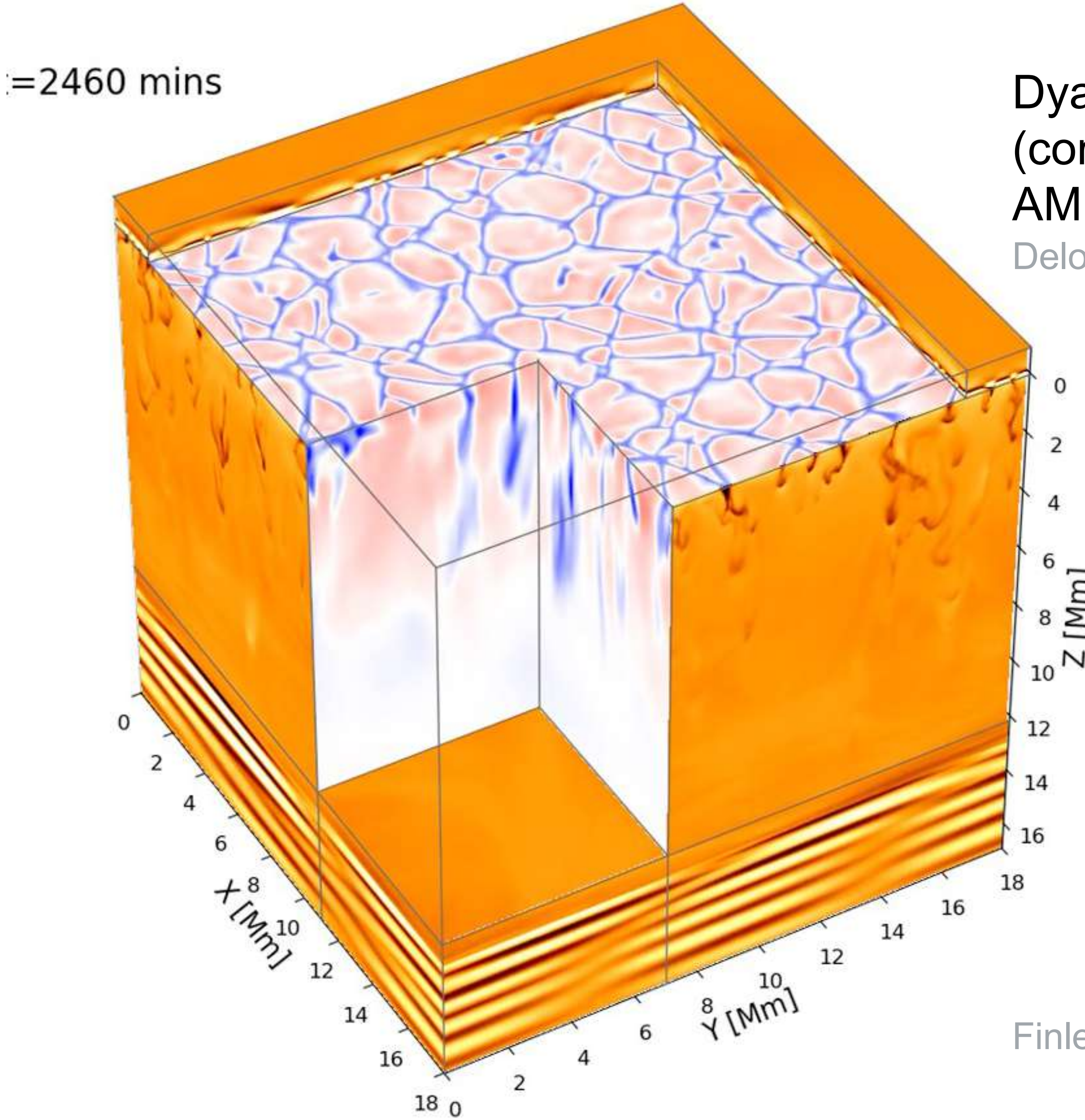
∇_e and ∇_{ad} are similar in nature since both describe temperature variation of a gas undegoing a pressure change. Whereas. ∇_{rad} and ∇_μ describe a spatial variation of T and μ of the medium .

Typical convection velocity in solar-type stars (M=[1,1.6]M_{Sun})



Brun et al. 2026

3-D advanced turbulent convection ('sandwiched' between 2 stable zones)



Dyablo Whole Sun code (compressible, MHD, AMR, gpus)

Delorme, Brun et al. 2024

https://wholesun.eu

Finley, Brun et al. 2026 in prep

Stability Criterion of Schwartzschild and Ledoux

Consider an atmosphere in which énergy is transported by radiation (or conduction) only. Then $\nabla = \nabla_{rad}$. Let's test the stability of this atmosphere by displacing adiabatically a small element: $\nabla_e = \nabla_{ad}$

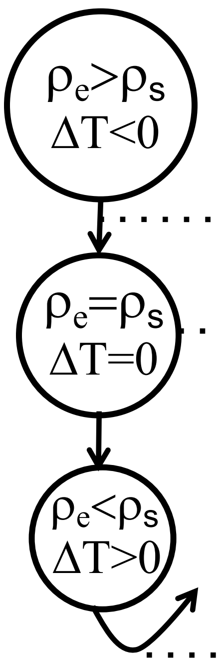
Following previous derivation the atmosphere is stable if:

Ledoux criterion $$\nabla_{rad} < \nabla_{ad} + \frac{\phi}{\delta}\nabla_{\mu}$$

Ideal gas: P = R ρ T/ μ
=> α=δ=φ=1

If there is no variation of composition nor ionization states then:

Schwarzschild criterion $$\nabla_{rad} < \nabla_{ad}$$



$F_{conv} > 0$

$\nabla = \nabla_{ad}$ at $r = r_{bcz}$

$r = r_f$ $F_{conv} = 0$

$\nabla < \nabla_{ad}$

$F_{conv} < 0$

$r = r_{ov}$ $F_{conv} = 0$

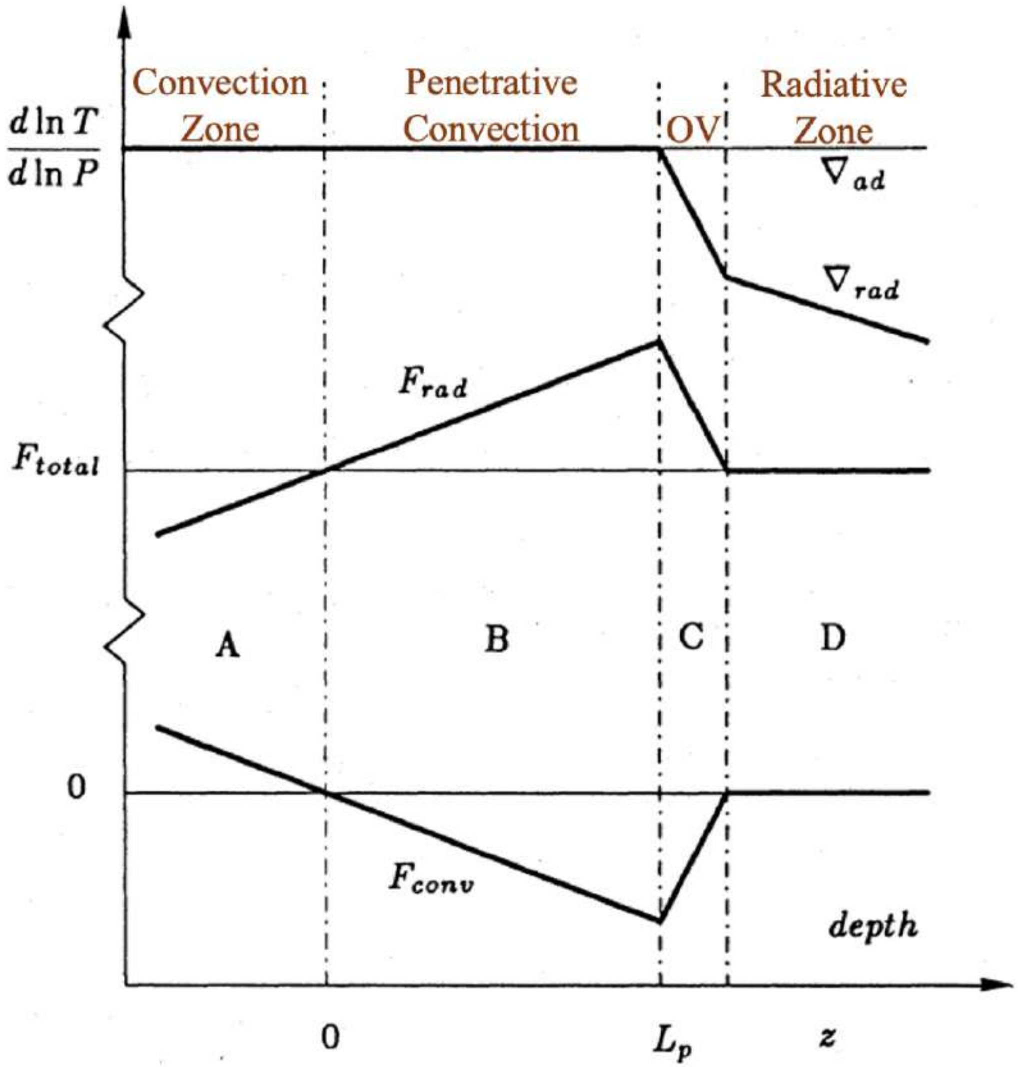
Overshooting vs Penetrative Convection

Péclet number: Pe= vL/κ
Pe >> 1 (Sun) -> change stratification
Pe ~1 , extended overshoot

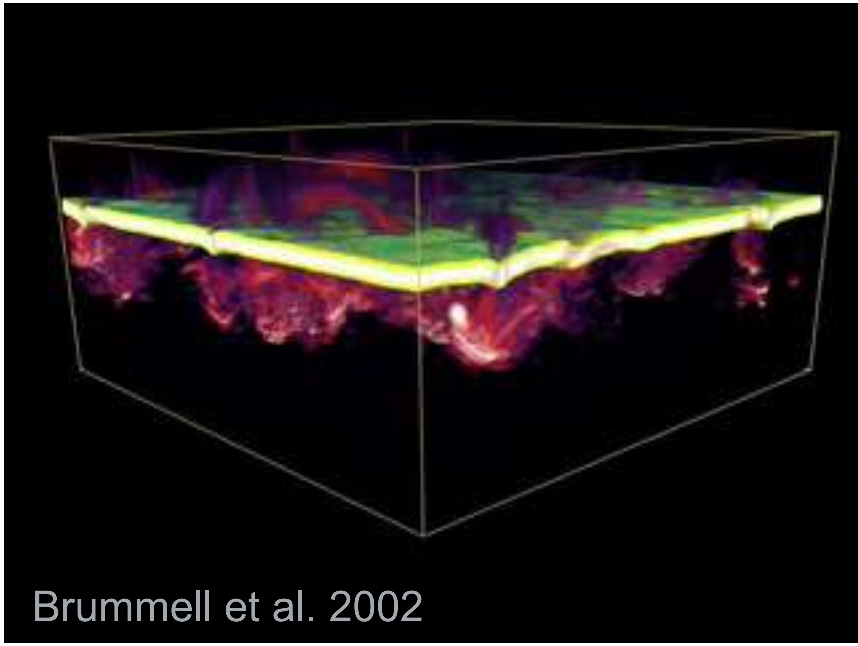
If a heavy sinking convective parcel penetrates into a subadiabatic layer, when the temperature fluct changes sign, parcel is neutrally buoyant but continues to sink by virtue of inertia until the Buoyancy force reverses the direction of its motion.

Zahn 1991

Stability Criterion of Schwartzschild and Ledoux



Zahn 1991



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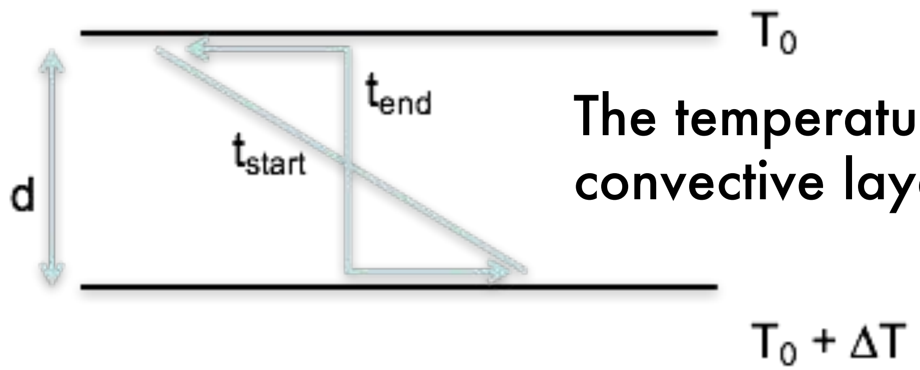
If a heavy sinking convective parcel penetrates into a subadiabatic layer, when the temperature fluct changes sign, parcel is neutrally buoyant but continues to sink by virtue of inertia until the Buoyancy force reverses the direction of its motion.

Convection and Rayleigh number (non inviscid convection case)



The temperature gradient is homogenized in a convective layer, creating boundary layers.

Note: here $\delta\rho = \rho_e - \rho_s$

A plasma bubble less dense than its surroundings rises at a speed w sufficient to counteract viscous friction:

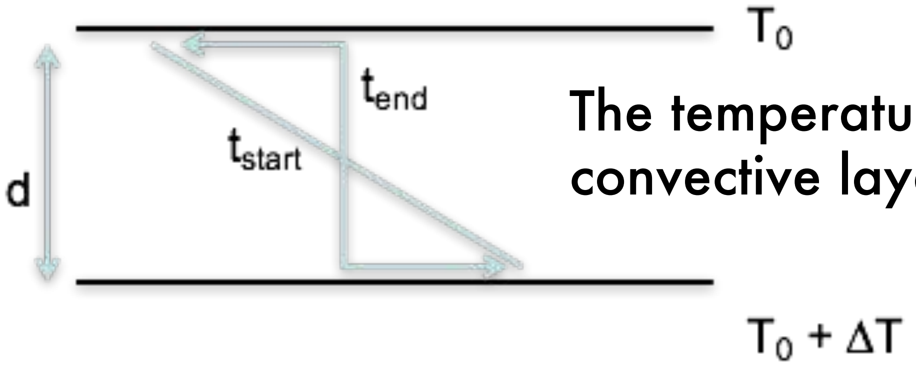
$$g\delta\rho \sim \mu\Delta w \sim \rho\nu w/d^2 \longrightarrow w = \delta\rho g d^2/\nu\rho$$

with $\mu = \rho\nu$ the dynamic viscosity and ν the kinematic viscosity.

For an ideal gas, density disturbances can be related to a temperature variation ΔT by a thermal expansion factor α, such that $\delta\rho/\rho=\alpha\Delta T$, and the blob velocity becomes:

$$w \sim \alpha\Delta T g d^2/\nu$$

Convection and Rayleigh number (non inviscid convection case)



The temperature gradient is homogenized in a convective layer, creating boundary layers.

For an ideal gas, density disturbances can be related to a temperature variation ΔT by a thermal expansion factor α, such that $\delta\rho/\rho=\alpha\Delta T$:

$$w \sim \alpha \Delta T g d^2 / \nu$$

As the bubble rises, it loses some of its heat through radiation. Its rate of ascent must therefore be sufficiently rapid, i.e., $d/w < d^2/\kappa$ we then get:

$$1 < \alpha \Delta T g d^3 / \nu\kappa = \mathrm{Ra}$$

Hence, the Rayleigh number must be larger than 1 in this back of the envelope derivation

Rayleigh-Bénard convection

Plane layer conv (cf. Chandrasekhar's book in 1961)

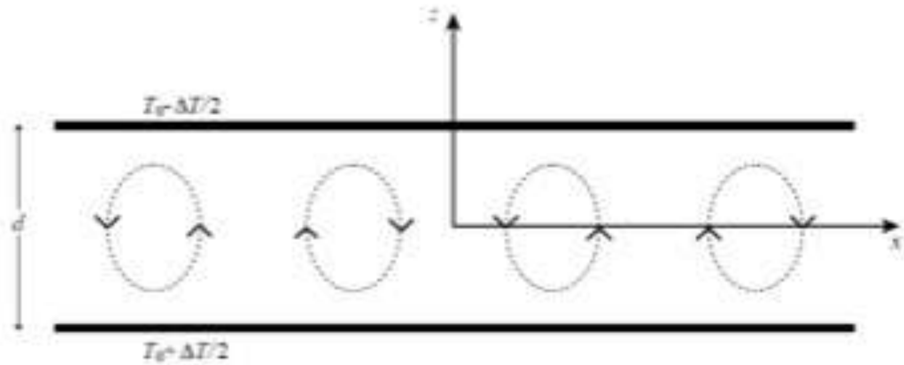


Fig. 17.1: Rayleigh-Bernard convection. A fluid is confined between two horizontal surfaces separated by a vertical distance d. When the temperature difference between the two plates ΔT is increased sufficiently, the fluid will start to convect heat vertically. The reference effective pressure P_0' and reference temperature T_0 are the values of P' and T measured at the midplane $z = 0$.

Rayleigh Number:

$$Ra = \frac{g\alpha\Delta T d^3}{\kappa\nu}$$

If Ra is sufficiently large then convection starts. The main difference with Schwatzchild criterion comes from considering diffusion effects

Bounday conditions: stress-free top & bottom: $Ra_c = 658$

stress-free top & no slip bottom: $Ra_c = 1100$

no slip top & bottom : $Ra_c = 1708$

With a vertical magnetic field pervading the system:
BC's stress free top & bottom for V, radial field BC's for B: Ra_c depends on Hartman number, i.e Ha >> 1, alors $Ra_c = \pi^2(Ha)^2$

$$Ha = \left(\frac{\sigma B_0^2 d^2}{\rho\nu}\right)$$

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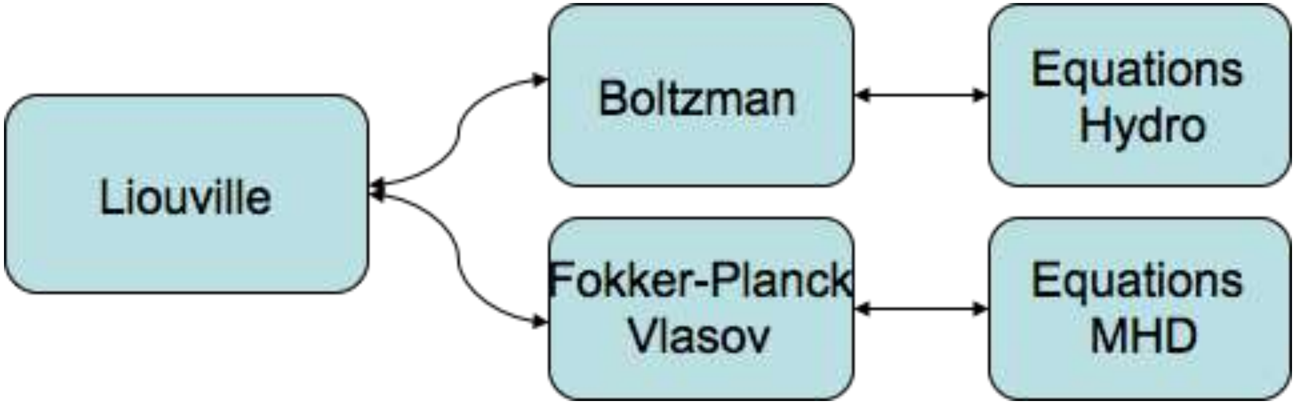
Derivation of fluides and plasmas equations of motion

Fluids and gases are made up of atoms and molecules: they can be described either as a continuum according to macroscopic equations (the "fluid" approach), or as a collection of interacting particles (the "particles" approach).

We wish to develop a dynamical theory of fluid and plasmas, that is:
1) We need a method for describing the state of the system based on a set of variables.
2) We need a set of equations giving the time derivative of these variables.



Hierarchy of different theories (I)

Level	Neutral fluids	Plasmas
0	N particules quantiques $\Psi(\vec{x}_1, ..., \vec{x}_N)$ Eq de Schrödinger	idem fluides neutres
1	N particules classiques $(\vec{x}_1, ..., \vec{x}_N; \vec{u}_1, ..., \vec{u}_N)$ Eq de Liouville Lois de Newton (eq d' Hamilton)	Idem fluides neutres
2	Fonction de distribution $f(\vec{x}, \vec{u}, t)$ Eq de Boltzmann	Fonction de distribution $f(\vec{x}, \vec{u}, t)$ Eq de Vlasov, Fokker-Planck
2 1/2	N/A	Modèle à deux fluides ions (massifs) et électrons $f_i(\vec{x}, \vec{u}, t), f_e(\vec{x}, \vec{u}, t)$
3	Modèle continu $\rho(\vec{x}, t), T(\vec{x}, t), \vec{u}(\vec{x}, t)$ Eq Hydrodynamiques	Modèle a un fluide $\rho(\vec{x}, t), T(\vec{x}, t),$ $\vec{u}(\vec{x}, t), \vec{B}(\vec{x}, t)$ Eq Magnétohydrodynamiques

Hierarchy of different theories (II)

Level	Neutral fluids	Plasmas
0	N particules quantiques $\Psi(\vec{x}_1, ..., \vec{x}_N)$ Eq de Schrödinger	idem fluides neutres
1	N particules classiques $(\vec{x}_1, ..., \vec{x}_N; \vec{u}_1, ..., \vec{u}_N)$ Eq de Liouville Lois de Newton (eq d' Hamilton)	Idem fluides neutres

Can we always go from level 0 to 1?

No if the quantum effects are important, *cf* equation of states of white dwarfs in the previous lecture. We need to avoid overlap of the wave packets, defining their charactistic size (wavelenght) as:

$$\lambda = h/p \sim h/\sqrt{mk_BT}$$

Ehrenfest theorem: if $hn^{1/3}/\sqrt{mk_bT} \ll 1$ then quantum effects can be neglected (n is the particle number density).

Hierarchy of different theories (III)

1	N particules classiques $(\vec{x}_1, ..., \vec{x}_N; \vec{u}_1, ..., \vec{u}_N)$ Eq de Liouville Lois de Newton (eq d' Hamilton)	Idem fluides neutres
2	Fonction de distribution $f(\vec{x}, \vec{u}, t)$ Eq de Boltzmann	Fonction de distribution $f(\vec{x}, \vec{u}, t)$ Eq de Vlasov, Fokker-Planck

Hamiltonian

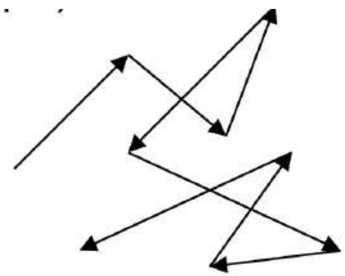
$$\partial_t f_s + [H, f_s] = 0$$

Vlasov equation

Fonction de distribution

$$\partial_t f_s + [H, f_s] = \mathcal{C}(f)$$

Boltzmann equation (includes collisions)



If the Knudsen number $\lambda_{mfp}/L << 1$, then we can use an approach where we work with average values describing the entire distribution function.

Hierarchy of different theories (IV)

We can then integrate the Vlasov/Boltzmann equation over momentum space to obtain a series of equations known as the "momentum" equations.

$$\int f \mathrm{d}u = n \longrightarrow \frac{\partial}{\partial t}(nm) + \frac{\partial}{\partial x_i}(nm < u_i >) = 0$$

$$<=> \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \text{ , avec } \rho = nm \text{ et } v_i =< u_i >$$

with $\int f u_i \mathrm{d}u = n \langle u_i \rangle$. Then we continue by integrating Vlasov against 'v' to get the equation of momentum conservation, and then against '(v-u)^2' to get energy conservation, etc...

$$\rho \left(\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} \right) = -\frac{\partial P_{ij}}{\partial x_i} + \frac{\rho}{m} F_j$$ Momentum conservation

$$\rho \left(\frac{\partial \epsilon}{\partial t} + v_i \frac{\partial \epsilon}{\partial x_i} \right) + \frac{\partial q_i}{\partial x_i} + P_{ij}\Lambda_{ij} = 0$$ Energy conservation

$$\epsilon = 0.5 < |u - v|^2 > \quad , P_{ij} = nm < (u_i - v_i)(u_j - v_j) >= nm[< u_i u_j > - v_i v_j]$$

$$\vec{q} =< (\vec{u} - \vec{v})|\vec{u} - \vec{v}|^2 > \quad , \Lambda_{ij} = 0.5 \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

Hierarchy of different theories (V): Euler and Navier-Stokes

Collisions play an important role to establish a fluid behavior and one can expect that The distribution are Maxwellian

$$P_{ij} = p\delta_{ij} \qquad p = nk_BT \;\;, q = 0 \qquad P_{ij}\Lambda_{ij} = 0.5p\delta_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = p\vec{\nabla}\cdot\vec{v}$$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

Nombre de Knudsen
K = λ/L, si K <<1 alors
approche fluide valide

$$\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\rho}{m}F_j \;, \quad \rho\left(\frac{\partial \epsilon}{\partial t} + v_i\frac{\partial \epsilon}{\partial x_i}\right) + p\vec{\nabla}\cdot\vec{v} = 0$$

Note: 5 var=5 eq, we have achieved our goal of having a dynamical theory. However q=0 (no heat flux) and Pij is diagonal, so no viscosity, it is not ideal… To improve, it is necessary to consider departure from Maxwellian distribution in order to develop a theory of transport processes:

$$P_{ij} = p\delta_{ij} + \pi_{ij} \qquad \pi_{ij} = -2\mu[\Lambda_{ij}\Lambda_{ij} - 1/3(\vec{\nabla}\cdot\vec{v})^2]$$

(cas μ cst)

$$\vec{q} = -K\vec{\nabla}T$$

More realistic case

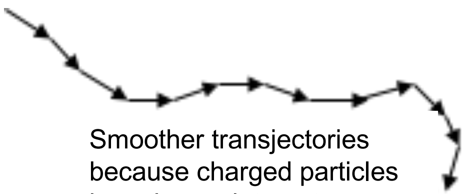
$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

$$\rho\left(\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_i} + \mu\left[\nabla^2 v_j + 1/3\frac{\partial}{\partial x_j}(\vec{\nabla}\cdot\vec{v})\right] + \frac{\rho}{m}F_j$$

$$\rho\left(\frac{\partial \epsilon}{\partial t} + v_i\frac{\partial \epsilon}{\partial x_i}\right) - \vec{\nabla}\cdot(K\vec{\nabla}T) + p\vec{\nabla}\cdot\vec{v} - 2\mu[\Lambda_{ij}\Lambda_{ij} - 1/3(\vec{\nabla}\cdot\vec{v})^2] = 0$$

Hierarchy of different theories (VI): plasma case

For plasma, the interaction between charged particles is the Coulomb interaction, which significantly changes the fluid behavior of the system.

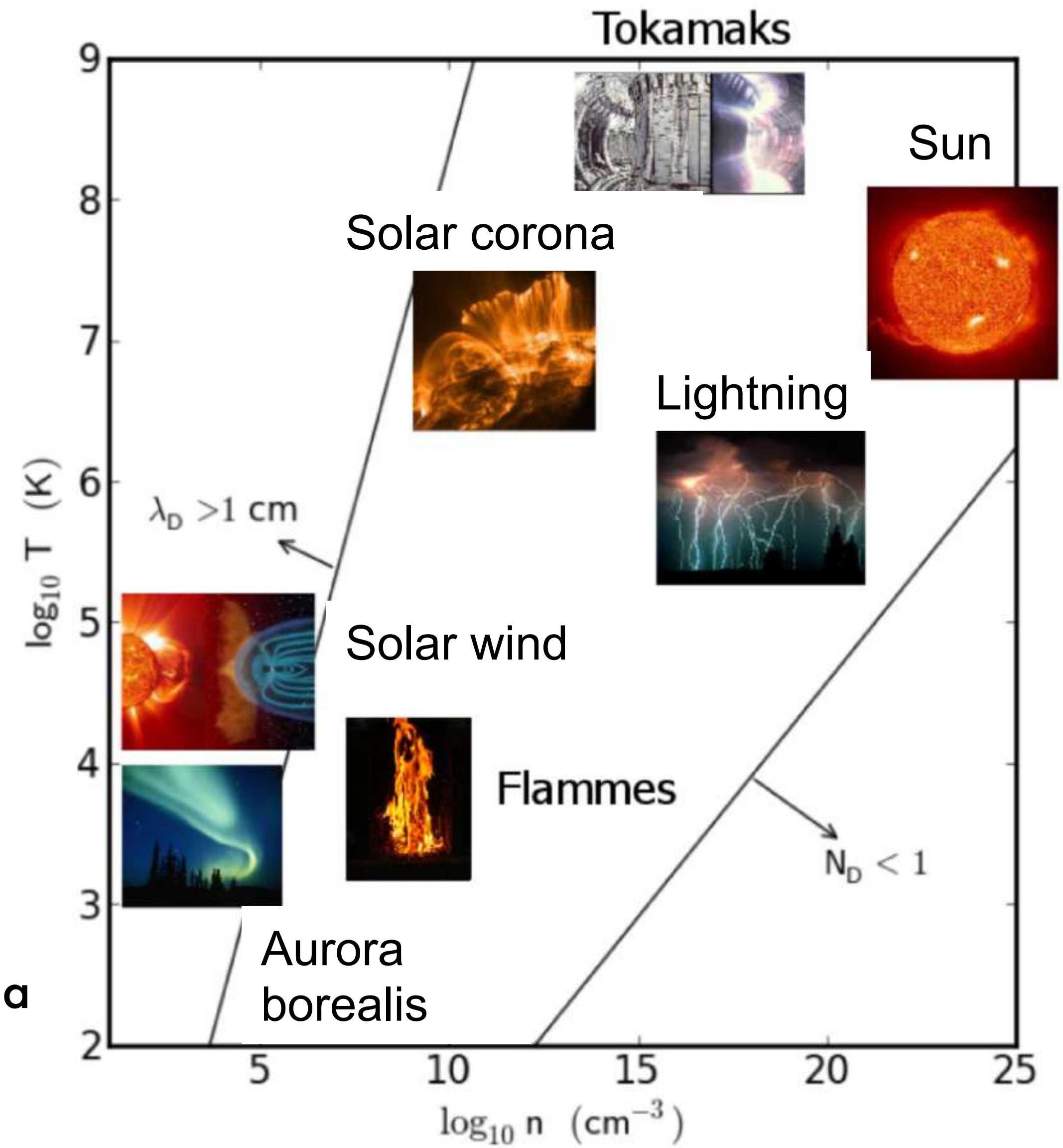


Smoother transjectories because charged particles in a plasma interact on Average from further away so weaker effects

The key parameter for a plasma is **Debye lenght**

$$\lambda_D = \left(\frac{k_B T}{4\pi n e^2}\right)^{1/2}$$

This is the length beyond which plasma can be considered neutral (screening of distant particle charges).



Hierarchy of different theories (VII)

Let the distribution functions fi(**x**,**u**,t) and fe(**x**,**u**,t) representing the ions and electrons of a plasma and let's make the hypothesis that they all follow Vlasov equation. If the force F is purely electromagnetic in nature, then Vlasov equation reads:

$$\frac{\partial f_\ell}{\partial t} + \vec{u}\cdot\vec{\nabla} f_\ell + \frac{q_\ell}{m_\ell}(\vec{E} + \frac{\vec{u}}{c}\times\vec{B})\cdot\vec{\nabla}_{\vec{u}} f_\ell = 0$$

With l=i or e and qe=e and qi=-e, and me and mi respectively the electons and ions mass. Let's multiply by χ(u) and integrate

$$\frac{\partial}{\partial t} n_\ell < \chi > + \frac{\partial}{\partial x_i} n_\ell < \chi u_i > - \frac{n_\ell}{m_\ell} < \frac{\partial \chi}{\partial u_i} F_i >= 0$$

If χ=1 then we obtain the conitnuity equation:
With $\vec{v}_\ell =< \vec{u} >_\ell$ the average velocity of particle type i.

$$\frac{\partial n_\ell}{\partial t} + \nabla\cdot(n_\ell \vec{v}_\ell) = 0$$

If $\chi = m_i u_i$, we obtain after some manipulations the Euler equation for ions and e-
(considering the pressure as a scalar)

$$m_\ell n_\ell \left(\frac{\partial \vec{v}_\ell}{\partial t} + (\vec{v}_\ell\cdot\nabla)\vec{v}_\ell\right) = -\nabla p_\ell + q_\ell n_\ell \left(\vec{E} + \frac{\vec{v}_\ell}{c}\times\vec{B}\right)$$

16 scalars
10 equations

cold plasma approxim.

$$p_\ell = 0$$

T_i & T_e cst

$$p_\ell = n_\ell k_B T_\ell$$

$$p_\ell = C n_\ell^\gamma$$

Processus adiabatiques

équations de Maxwell => + 6 équations

$$\vec{\nabla}\cdot\vec{E} = 4\pi(n_i - n_e)e, \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t},$$

$$\vec{\nabla}\cdot\vec{B} = 0, \ \vec{\nabla}\times\mathbf{B} = \frac{4\pi}{c}(n_i\vec{v}_i - n_e\vec{v}_e)e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

16 variables = 16 equations ⇔ théorie dynamique Two fluids model

Valid for collisionless plasma

Hierarchy of different theories (VIII): induction equation

If we consider phenomena on spatial scales larger than λ_D and temporal scales longer than the plasma frequency $\omega_D = 4\pi Ze/m^{1/2}$, then the plasma can be treated as a single fluid.

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (4)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad (5)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \quad (7)$$

From Maxwell equartion (5) and (7), and by neglecting the displacement current (if v << c):

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \text{ et } \mathbf{J} = \frac{c}{4\pi}(\nabla \times \mathbf{B}),$$

Assuming Ohm's law for a conducting fluid (plasma) in a motion at speed v: $\mathbf{J} = \sigma\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right)$

One can derive the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} = -\nabla \times \left(\frac{c\mathbf{J}}{\sigma} - \mathbf{v} \times \mathbf{B}\right)$$
$$= -\nabla \times \left(\frac{c^2}{4\pi\sigma}\nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B}\right)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta\nabla \times \mathbf{B})$$

with $\eta = c^2/4\pi\sigma$ the magnetic diffusivity

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta\Delta\mathbf{B}, \quad \text{if } \eta = cst.$$

Hierarchy of different theories (IX): MHD equations

Continuity, Navier-Stokes, Energie (+Lorentz force + Ohmic dissipation): :

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{v}), \quad \boxed{\nabla \cdot \mathbf{B} = 0} \\
\rho \frac{\partial \mathbf{v}}{\partial t} &= -\rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla P + \rho \mathbf{g} - 2\rho \boldsymbol{\Omega}_{\mathbf{o}} \times \mathbf{v} \\
&- \nabla \cdot \mathcal{D} + \boxed{\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}}, \\
\rho T \frac{\partial S}{\partial t} &= -\rho T(\mathbf{v} \cdot \nabla) S + \nabla \cdot (\kappa_r \rho c_p \nabla T) + \boxed{\frac{4\pi\eta}{c^2}\mathbf{J}^2} \\
&+ 2\rho\nu \left[e_{ij}e_{ij} - 1/3(\nabla \cdot \mathbf{v})^2\right] + \boxed{\rho\epsilon},
\end{aligned}
$$

Volumetric heat source (nuclear energy)

plus induction:

$$
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) \qquad (8)
$$

Lecture plan

I. Energy transport in stars

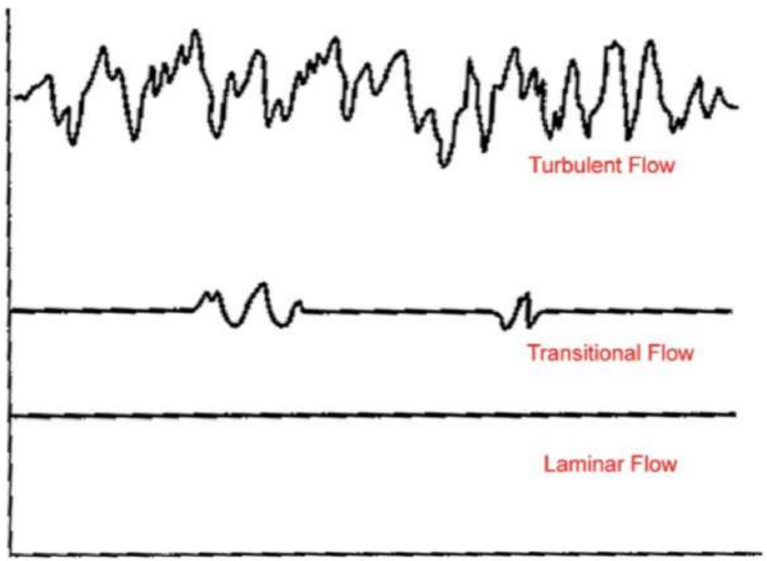
II. Stellar fluid dynamics
   From particle description to fluid description in plasmas
   Turbulence: basic concepts
   Convection simulation: some exemples

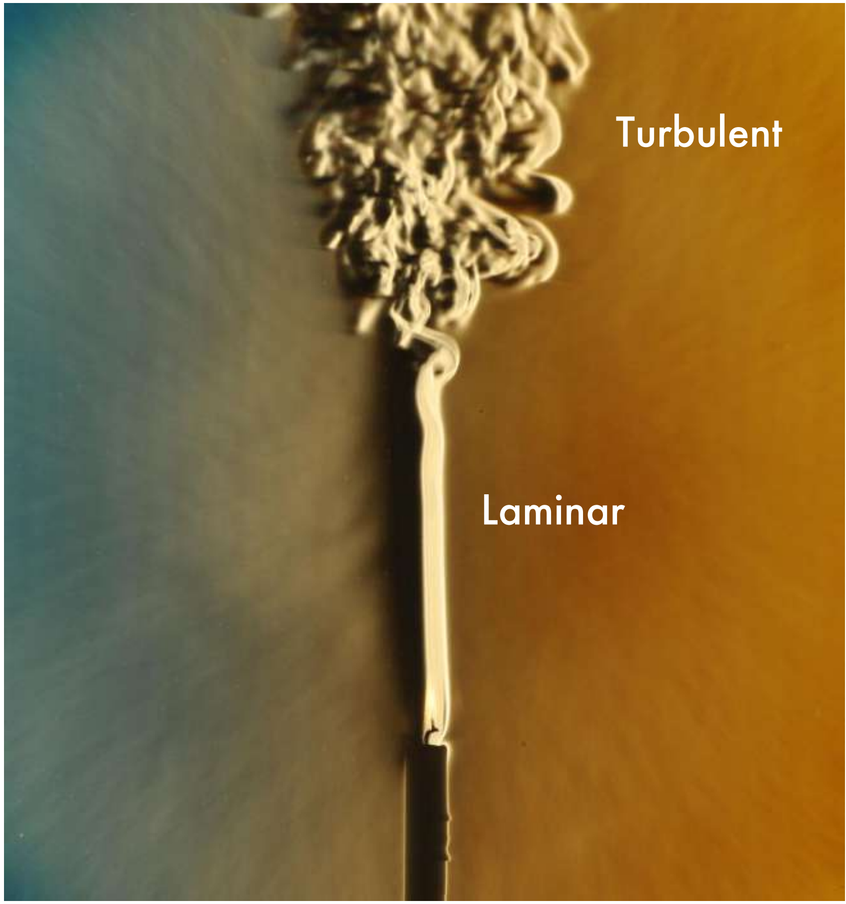
III. Dynamo effect in turbulent convective zones of stars
   The Sun's magnetic field
   The dynamo effect: fundamental ingredients
   The dynamo effect: kinematics vs. Dynamics
   Stellar magnetism and dynamo

Fluid Turbulence

Ordinary fluids are known to become turbulent when their Reynolds number Re=VL/ν (ν is the kinematic fluid viscosity) becomes large.

There are several types of turbulence, with different characteristics depending on the dimensionality of the turbulent system (2D, 3D) and the source of the turbulence (flow, density fluctuations, heat, plasma phenomena, etc.).



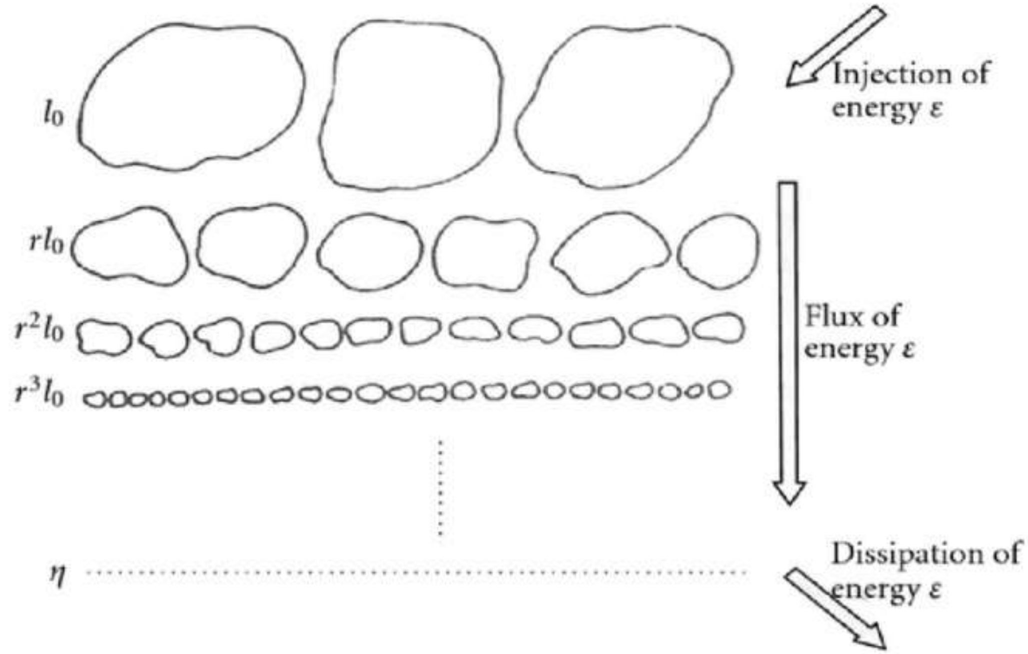


$$Re = \left[\frac{\rho(\mathbf{v}\cdot\nabla)\mathbf{v}}{\mu_{dyn}\Delta\mathbf{v}}\right] \sim \frac{\rho_0 V_0^2}{l_0}\frac{l_0^2}{\rho_0 \nu V_0} = \frac{V_0\, l_0}{\nu}$$

Fluid Turbulence (II)

Richardson Cascade
(1922)

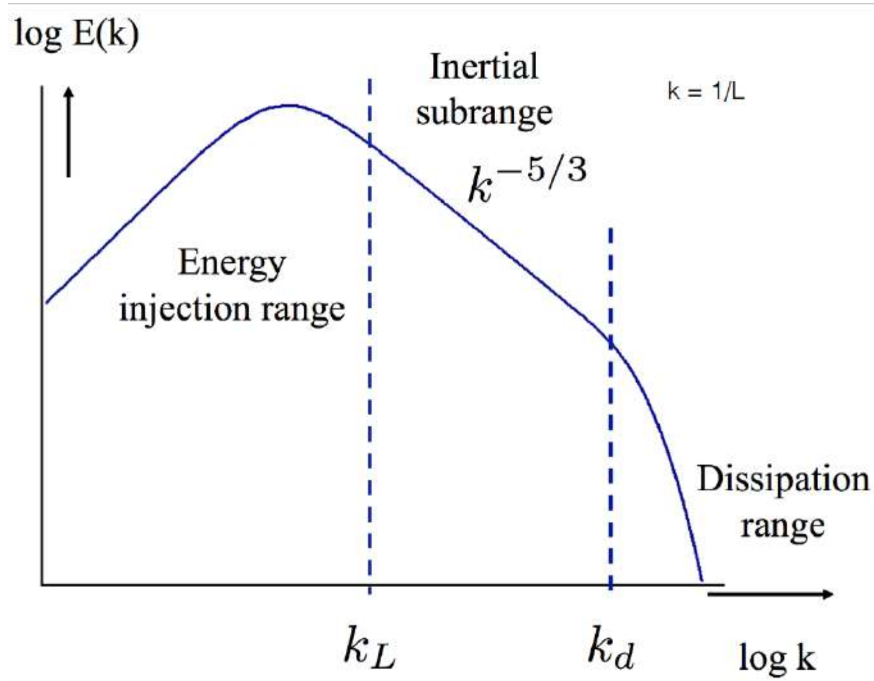
Big whorls have little whorls
Which feed on their velocity
And little whorls have lesser whorls
and so on to viscosity.



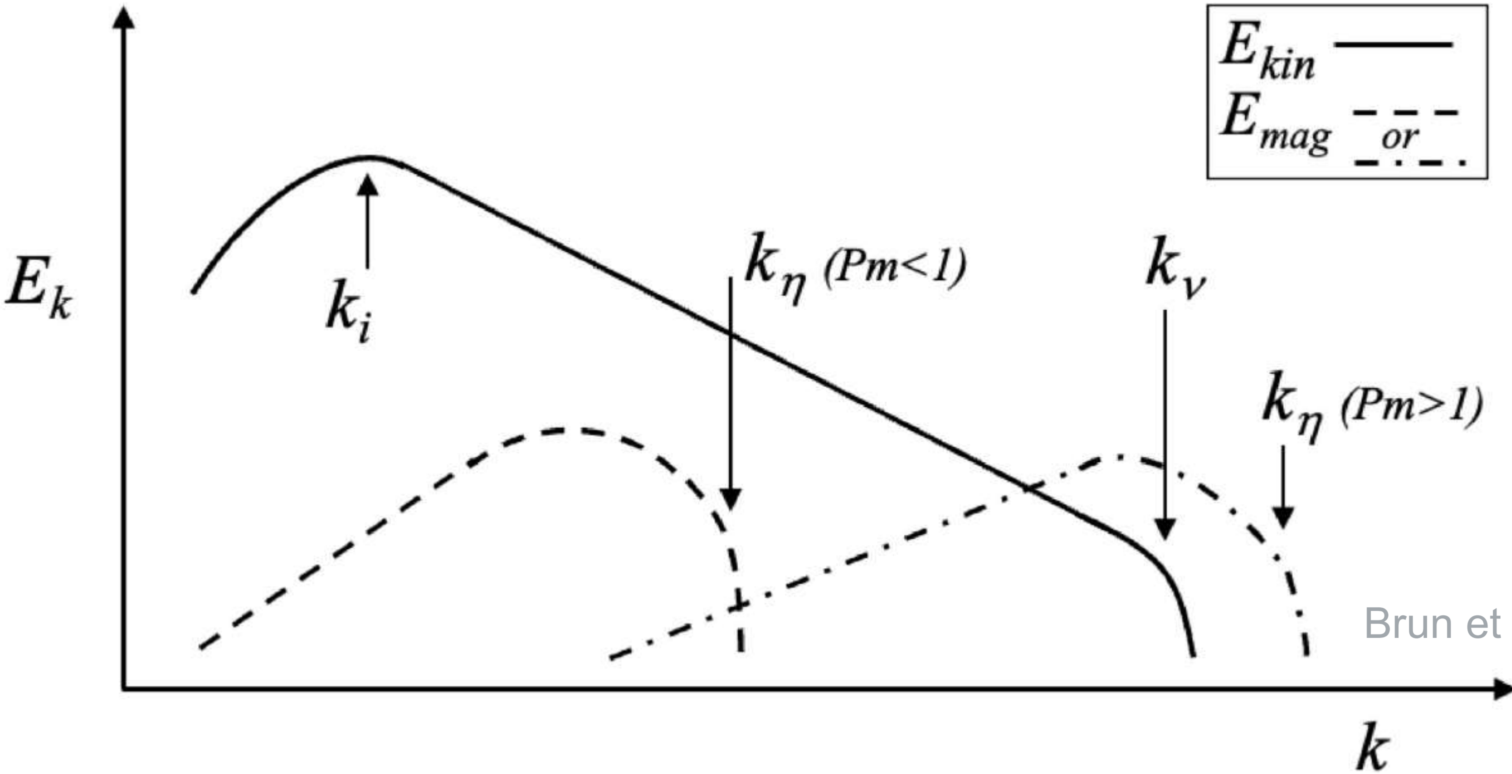
Kolmogorov phenomenology:

$$E_k \propto \varepsilon^{2/3} k^{-5/3}$$

See also Frisch 1995, Pope 2000



MHD Turbulence



Brun et al. 2026

MHD regimes Galtier 2020

"Strong": $E_k \propto (\varepsilon v_A)^{1/2} k^{-3/2}$

"Weak": $E_k \propto (\varepsilon/\tau_A)^{1/2} k^{-2}$

Magnetic Reynolds number

$$Rm = \left[\frac{\boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B})}{\eta \Delta \mathbf{B}}\right] \sim \frac{V_0 B_0}{l_0} \frac{l_0^2}{\eta B_0} = \frac{V_0 l_0}{\eta}$$

$$Rm = \frac{V_0 l_0}{\eta} = \frac{V_0 l_0}{\nu} \frac{\nu}{\eta} = Re\, Pm.$$

Total energy equation
(conservative form, compressible cas, stratified)

By dot product v . Navier-Stokes => kinetic energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \mathrm{div}\left(\frac{1}{2}\rho v^2\ \boldsymbol{v}\right) =$$

$$-\boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot\mathrm{div}\overline{\overline{\sigma}} + \rho\boldsymbol{v}\cdot\boldsymbol{g} + \ v.(jxB)$$

By dot product B . Induction => magnetic energy erquation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + U + \qquad + \frac{1}{2}\frac{B^2}{\mu_0}\right) +$$

$$\mathrm{div}\left(\frac{1}{2}\rho v^2\boldsymbol{v} + (U+P)\boldsymbol{v} + \boldsymbol{q} - \boldsymbol{v}\cdot\overline{\overline{\sigma}} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right) \quad = \rho v.g$$

Adding internal energy equation:

$$\partial U/\partial t + \mathrm{div}\,(U\ \ \boldsymbol{v} + \boldsymbol{q}) = H - P\mathrm{div}\boldsymbol{v} + (\overline{\overline{\sigma}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{v}$$

+ gravity term

$$\rho\boldsymbol{v}\cdot\boldsymbol{g} = -\rho\boldsymbol{v}\cdot\boldsymbol{\nabla}G = -(\mathrm{div}(\rho\boldsymbol{v}G) - G\mathrm{div}\rho\boldsymbol{v})$$

$$= -\mathrm{div}(\rho\boldsymbol{v}G) - G(\frac{\partial\rho}{\partial t}) = -\frac{\partial}{\partial t}(\rho G) - \mathrm{div}(\rho\boldsymbol{v}G)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + U + \qquad + \frac{1}{2}\frac{B^2}{\mu_0} + \rho G\right) +$$

Energy: kinetic, internal, magnetic, potential

$$\mathrm{div}\left(\frac{1}{2}\rho v^2\boldsymbol{v} + (U+P)\boldsymbol{v} + \boldsymbol{q} - \boldsymbol{v}\cdot\overline{\overline{\sigma}} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0} + \rho G\boldsymbol{v}\right) = H_{\neq joule}$$

Fluxes: kinetic energy, enthalpy, radiative, viscous, Poynting, gravity

Energy spectra and cascade (I)

The equations that describe the dynamics of an incompressible conducting fluid coupled to a magnetic field in the MHD approximation are given by

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f}, \qquad (1)$$

$$\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{b}, \qquad (2)$$

$$\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0, \qquad (3)$$

Simple case of imcompressible
Forced MHD turbulence

Let's use a Fourier representation

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \tilde{\mathbf{u}}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int \mathbf{u}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}^3$$

and

$$\mathbf{b}(\mathbf{x}) = \sum_{\mathbf{k}} \tilde{\mathbf{b}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \tilde{\mathbf{b}}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int \mathbf{b}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}^3,$$

and use a shell filter (mode is in [K,K+1]), scale K^{-1}

$$\mathbf{u}(\mathbf{x}) = \sum_K \mathbf{u}_K(x), \quad \mathbf{b}(\mathbf{x}) = \sum_K \mathbf{b}_K(x),$$

where

$$\mathbf{u}_K(x) = \sum_{K<|\mathbf{k}|\leq K+1} \tilde{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}},$$

and similarly for the field **b**,

$$\mathbf{b}_K(x) = \sum_{K<|\mathbf{k}|\leq K+1} \tilde{\mathbf{b}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Work by Pouquet, Minimi, Alexakis

Energy spectra and cascade (II)

The evolution of the kinetic energy in a shell K, $E_u(K)=\int \mathbf{u}_K^2/2dx^3$ is given by

$$\partial_t E_u(K) = \int \sum_Q [-\mathbf{u}_K \cdot (\mathbf{u}\cdot\nabla)\cdot \mathbf{u}_Q + \mathbf{u}_K \cdot (\mathbf{b}\cdot\nabla)\cdot \mathbf{b}_Q] - \nu|\nabla \mathbf{u}_K|^2 + \mathbf{f}\cdot \mathbf{u}_K d\mathbf{x}^3, \quad (4)$$

and for the magnetic energy $E_b(K)=\int \mathbf{b}_K^2/2dx^3$ we obtain

$$\partial_t E_b(K) = \int \sum_Q [-\mathbf{b}_K \cdot (\mathbf{u}\cdot\nabla)\cdot \mathbf{b}_Q + \mathbf{b}_K \cdot (\mathbf{b}\cdot\nabla)\cdot \mathbf{u}_Q] - \eta|\nabla \mathbf{b}_\mathbf{K}|^2 d\mathbf{x}^3. \quad (5)$$

Simple case of imcompressible Forced MHD turbulence

Work by Pouquet, Minimi, Alexakis

Compact notation

$$\partial_t E_u(K) = \sum_Q [\mathcal{T}_{uu}(Q,K) + \mathcal{T}_{bu}(Q,K)] - \nu\mathcal{D}_u(K) + \mathcal{F}(K), \quad (6)$$

$$\partial_t E_b(K) = \sum_Q [\mathcal{T}_{ub}(Q,K) + \mathcal{T}_{bb}(Q,K)] - \eta\mathcal{D}_b(K). \quad (7)$$

Here we have introduced the functions $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{ub}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, and $\mathcal{T}_{bu}(Q,K)$, which express the energy transfer between different fields and shells.

Energy spectra and cascade (III)

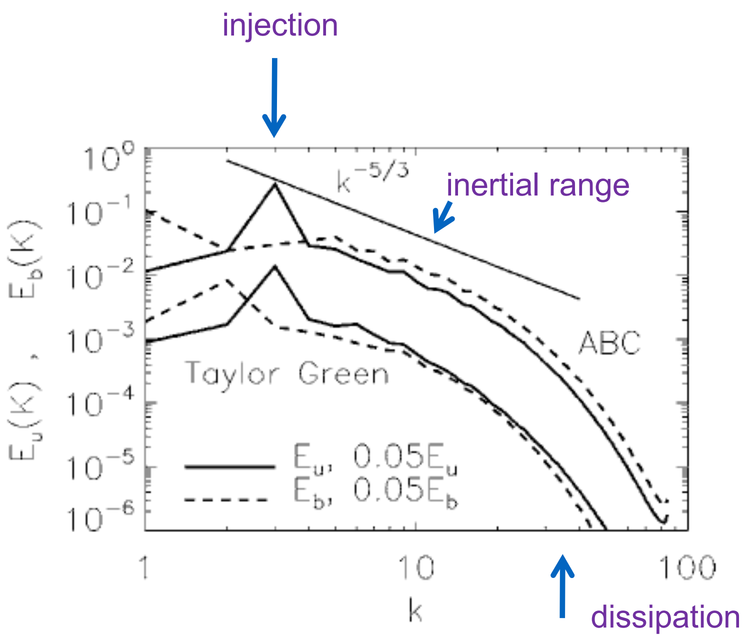


FIG. 1. Spectra of kinetic energy (solid line) and magnetic energy (dashed line) of the ABC and Taylor-Green runs, where the Taylor-Green spectra have been shifted down by a factor of 20 for clarity. The Kolmogorov slope is showed as a reference. Note that the magnetic Prandtl number $P_M \equiv \nu/\eta$ differs for the two runs.

Ponty et al. 2005

$$\partial_t E_u(K) = \sum_Q [\mathcal{T}_{uu}(Q,K) + \mathcal{T}_{bu}(Q,K)] - \nu \mathcal{D}_u(K) + \mathcal{F}(K), \quad (6)$$

Work by Pouquet, Minimi, Alexakis

$$\partial_t E_b(K) = \sum_Q [\mathcal{T}_{ub}(Q,K) + \mathcal{T}_{bb}(Q,K)] - \eta \mathcal{D}_b(K). \quad (7)$$

Here we have introduced the functions $\mathcal{T}_{uu}(Q,K)$, $\mathcal{T}_{ub}(Q,K)$, $\mathcal{T}_{bb}(Q,K)$, and $\mathcal{T}_{bu}(Q,K)$, which express the energy transfer between different fields and shells.

$$\mathcal{T}_{uu}(Q,K) \equiv -\int \mathbf{u_K}(\mathbf{u}\cdot\nabla)\mathbf{u_Q} d\mathbf{x}^3.$$

$$\mathcal{T}_{bu}(Q,K) \equiv \int \mathbf{u_K}(\mathbf{b}\cdot\nabla)\mathbf{b_Q} d\mathbf{x}^3.$$

$$\mathcal{T}_{ub}(Q,K) \equiv \int \mathbf{b_K}(\mathbf{b}\cdot\nabla)\mathbf{u_Q} d\mathbf{x}^3.$$

$$\mathcal{T}_{bb}(Q,K) \equiv -\int \mathbf{b_K}(\mathbf{u}\cdot\nabla)\mathbf{b_Q} d\mathbf{x}^3,$$

Triadic interactions of 2 wave numbers: A+B+C = 0 <=> A+B = -C
We recall that the sum of 2 vectors gives: A + B = +C

See also Yokoi 2013

Energy spectra and cascade (IV)

Table 1. Cascade directions of the ideal invariants comparing MHD and Navier–Stokes turbulence

	3-D		2-D	
MHD	E_k	direct	E_k	direct
	K_k	direct	K_k	direct
	H_k^M	inverse	H_k^ψ	inverse
Navier–Stokes	E_k^V	direct	E_k^V	inverse
	H_k^V	direct	Ω_k	direct

Biskamp 2003

$$E_k^V = 1/2\int_V u^2\,dV,\; E_k^M = 1/2\int_V B^2\,dV,\; E_k = 1/2\int_V (u^2+B^2)\,dV,$$

$$H_k^V = 1/2\int_V \mathbf{u}\cdot\boldsymbol{\omega}\,dV,\; H_k^M = 1/2\int_V \mathbf{A}\cdot\mathbf{B}\,dV,\; H_k = 1/2\int_V \mathbf{v}\cdot\mathbf{B}\,dV,$$

$$\Omega_k = 1/2\int_S (\nabla\times\boldsymbol{\omega})^2\,dS \text{ and } H_k^\Psi = 1/2\int_S \mathbf{A}^2\,dS.$$

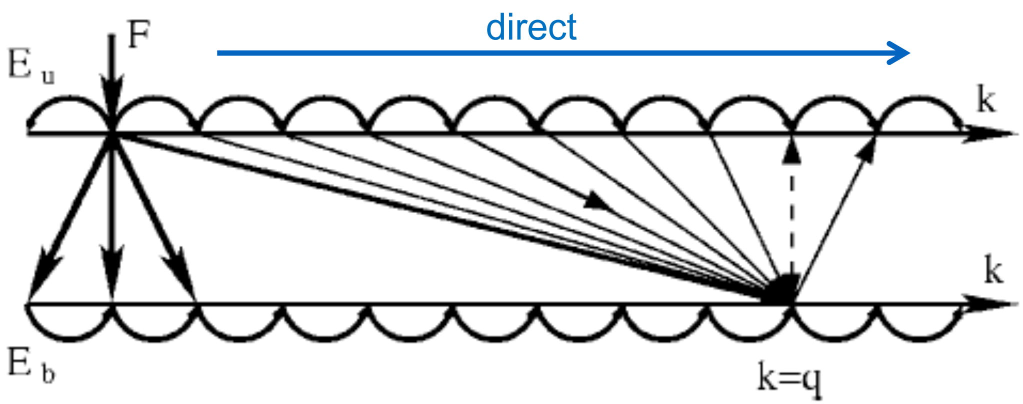


FIG. 14. A sketch of the energy transfer between different scales and different fields. The thickness of the lines is an indication of the magnitude of the transfers. The figure illustrates how energy is transferred to magnetic modes with wave number $k=q$ in the inertial range. The transfer between same fields is always local and direct. Each magnetic mode receives energy from all larger-scale velocity modes and gives to slightly smaller-scale velocity modes.

See also Yokoi 2013 on role of cross helicity

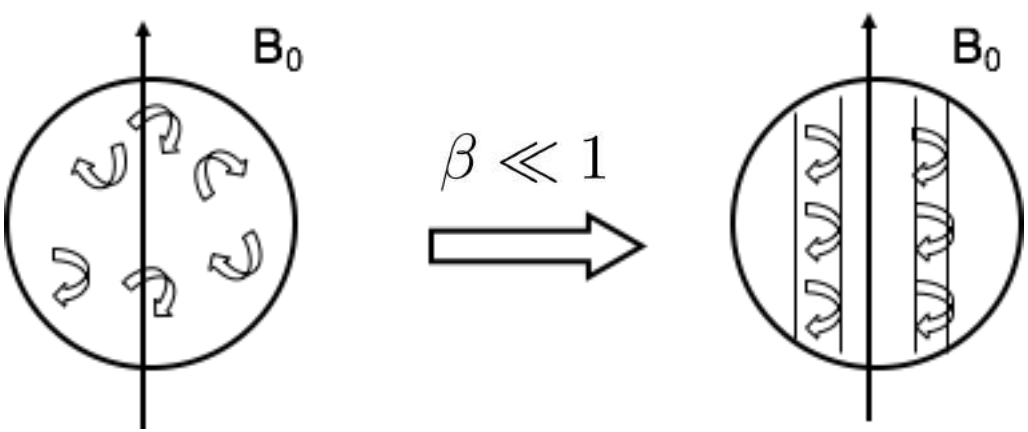
Remarks on turbulent MHD

Let's consider the Lundquist $S = Lv_A/\eta$ (Alfvén speed $v_A = B/(4\pi\rho)^{1/2}$)

and the magnetic Reynolds number $R_m = Lv/\eta$

S>1means that the resistance is low, but the system may well not be turbulent! It only becomes turbulent when Rm becomes large.
MHD turbulence is therefore found in highly dynamic systems such as tokamaks, solar eruptions, or solar convection.

$$\beta = \frac{8\pi n k_B T}{B^2} = \frac{\text{Thermal gas pressure}}{\text{Magnetic pressure}}$$



If β >> 1, since gas pressure dominates magnetic pressure (inside the Sun), the velocity field is more isotropic and fluctuates in all directions, not just perpendicular to the imposed field.

Lecture plan

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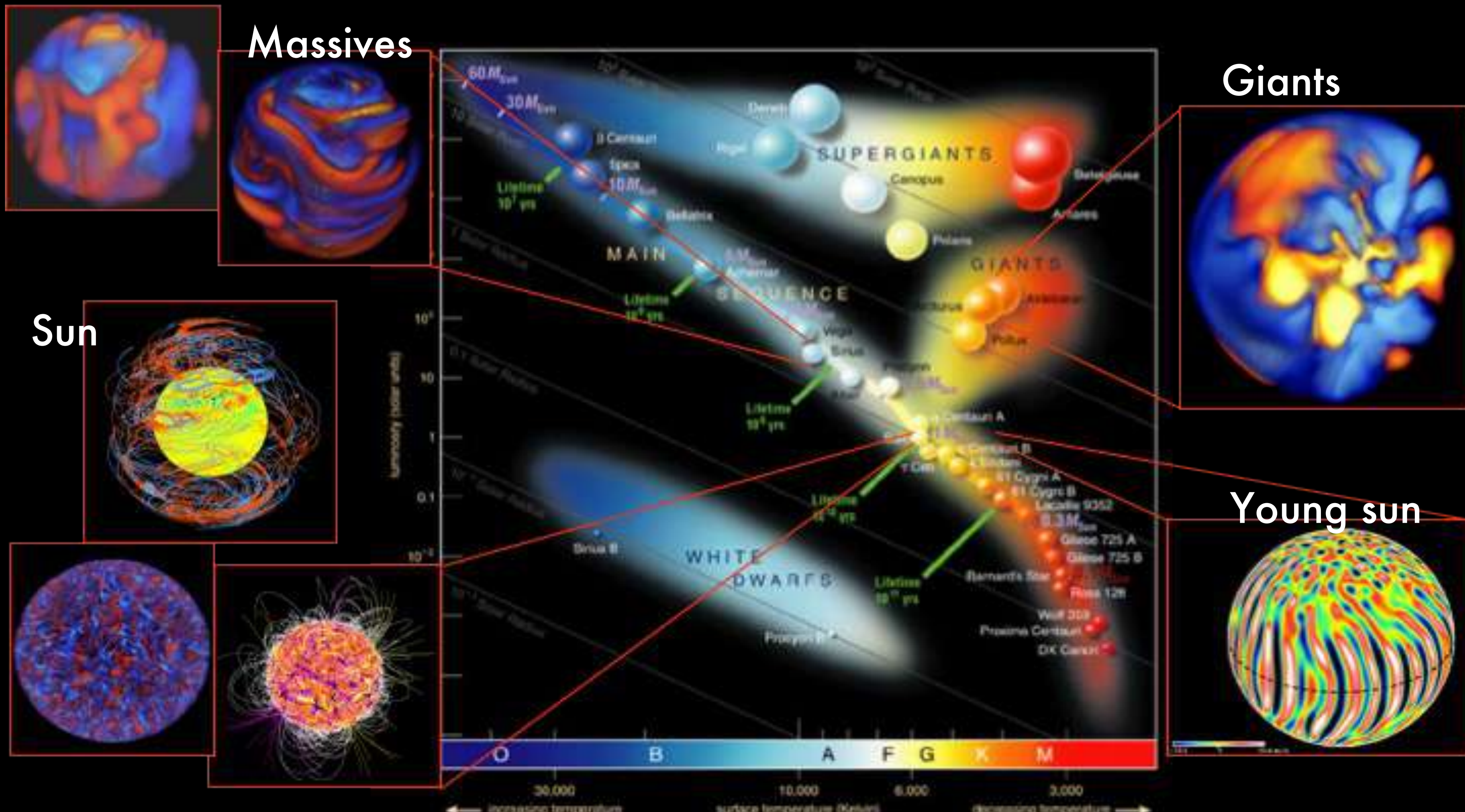
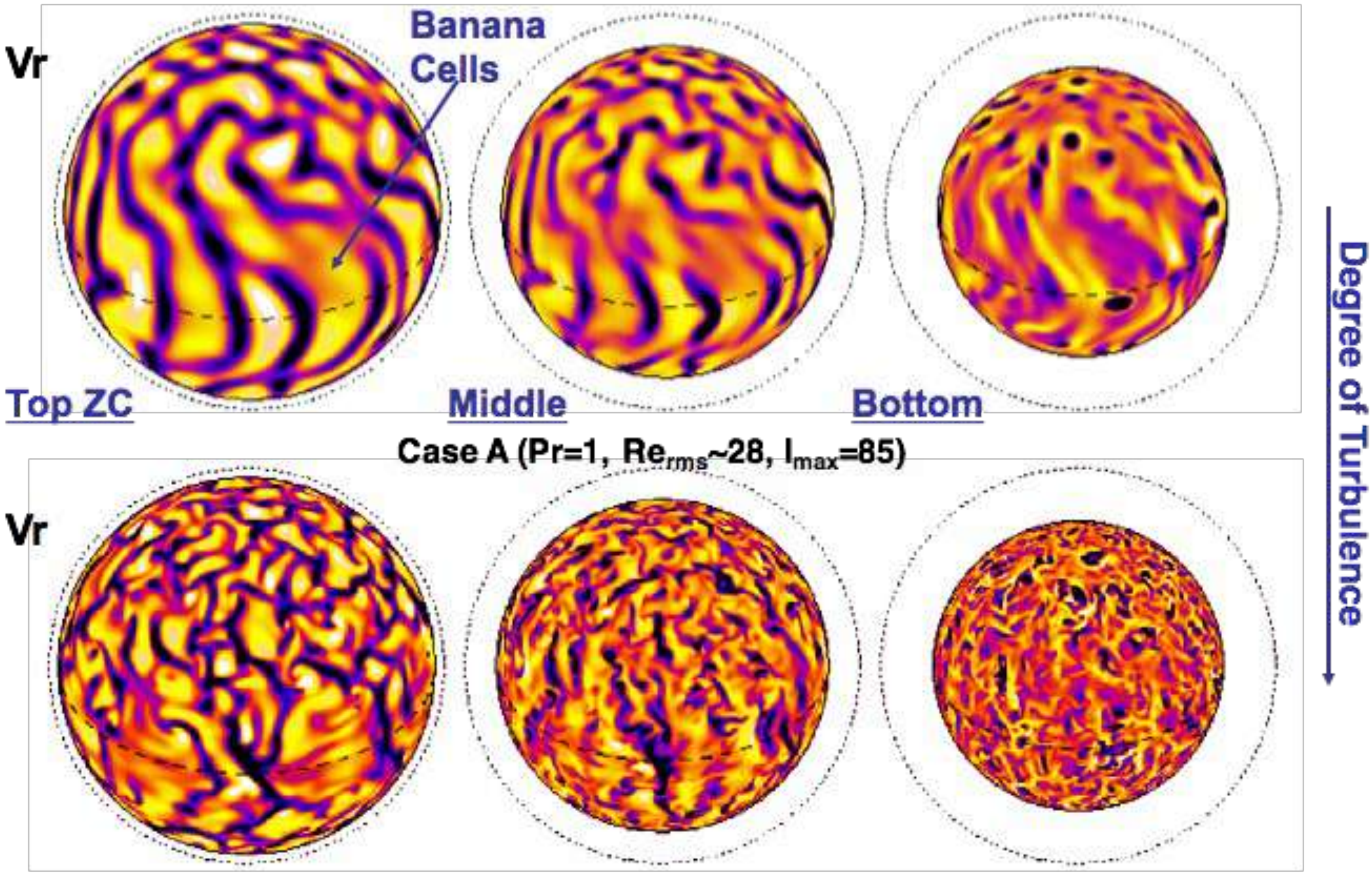
- Stellar magnetism and dynamo

Convection models in stars



Brun & Browning 2017
Brun et al. 2022

Quelques exemples de simulations modernes (I)



Case D (Pr=0.25, Re$_{rms}$~410, l$_{max}$=340)

Quelques exemples de simulations modernes (II)

Resolution~ 1500^3

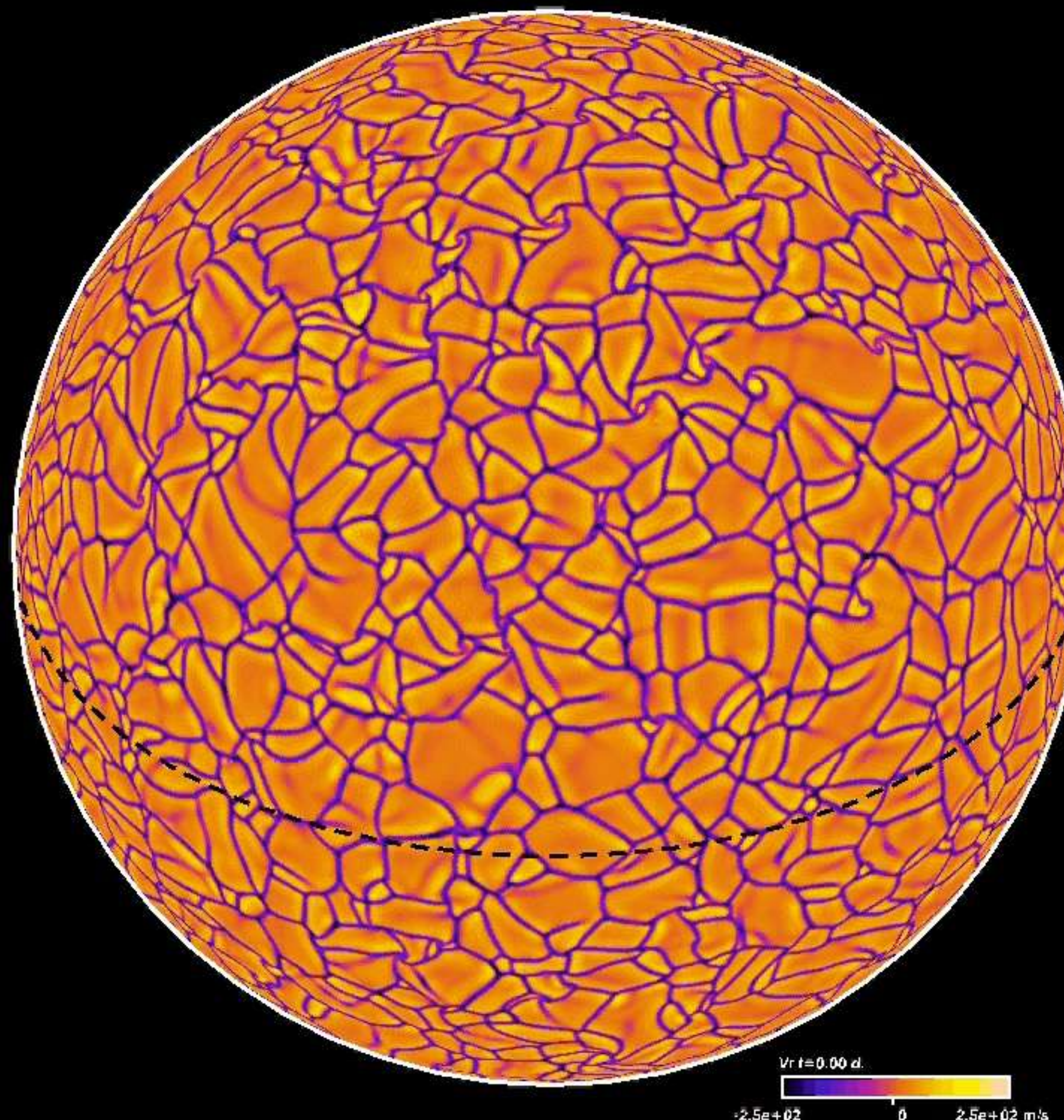
Re=VrmsD/ν~1000

Pr=0.25

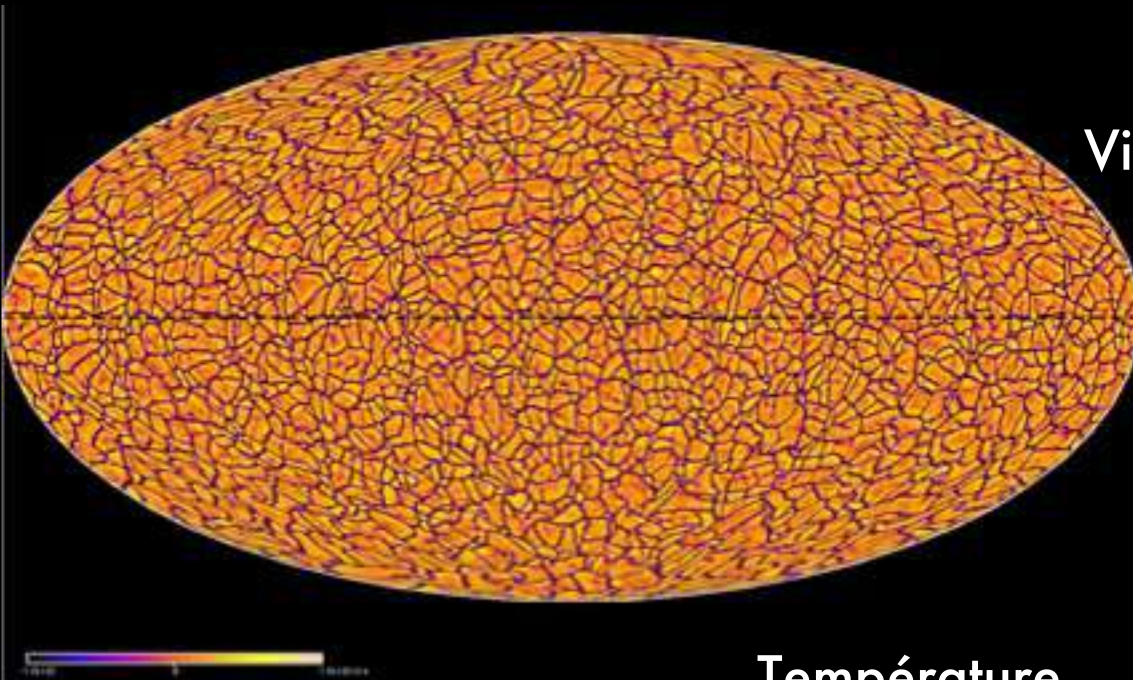
Simulation a 6000 cpus (BlueGene/p) Or 2000 BullX

depth=0.96 R

Brun 2011



Quelques exemples de simulations modernes (III)

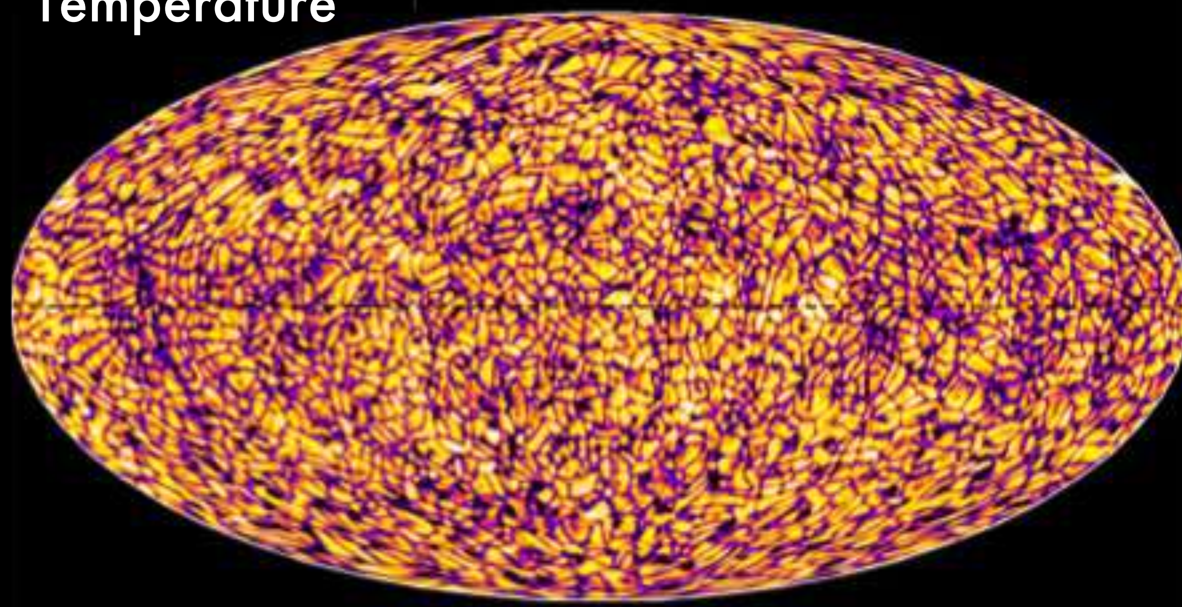


Vitesse radiale

Les corrélations <Vr T’> sont essentielles à la convection

(ce qui est chaud et léger monte)

Température

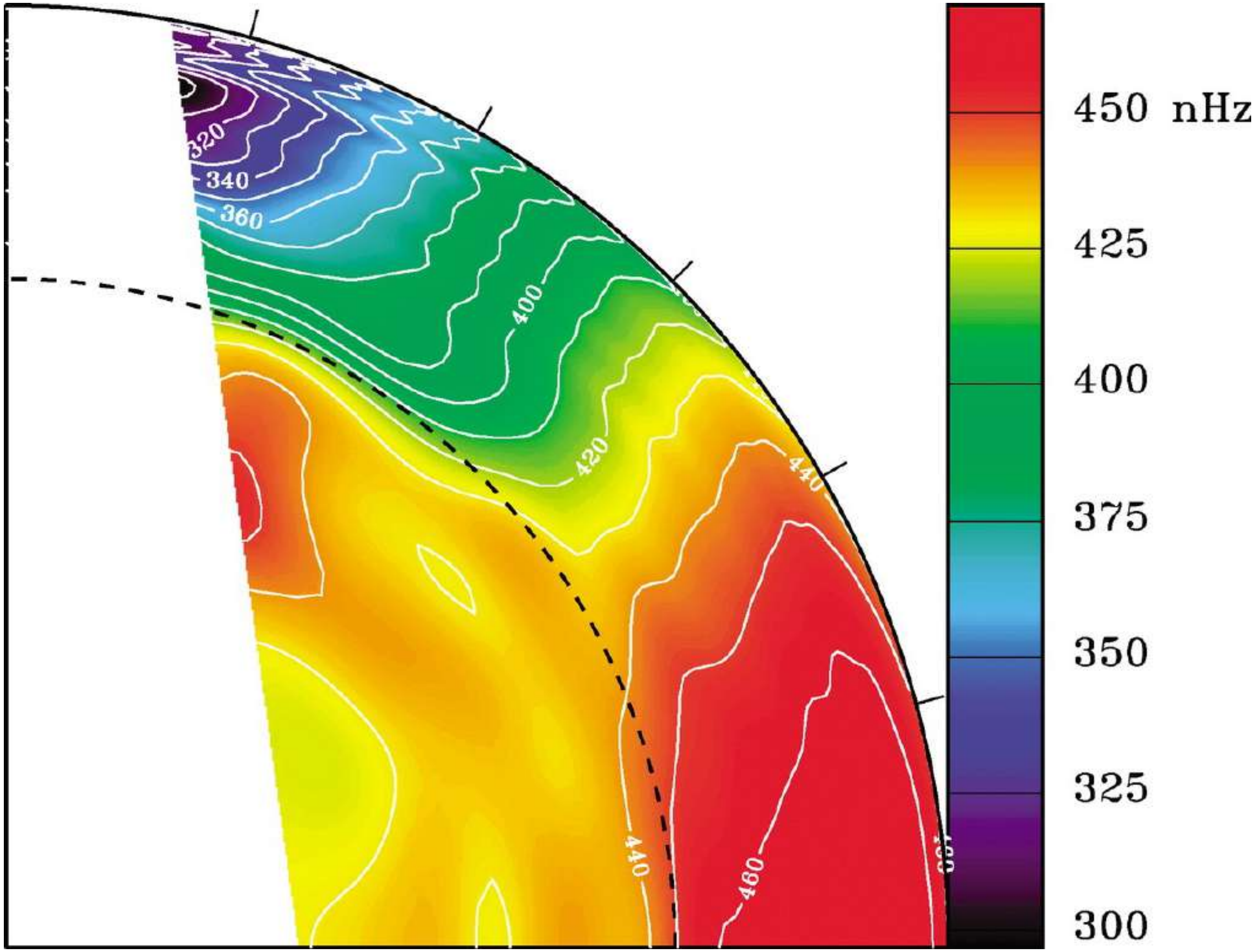


Resolution ~1800^3

Solar differential rotation

Helioseismology Inversion of the Solar internal angular velocity profiles (GONG)

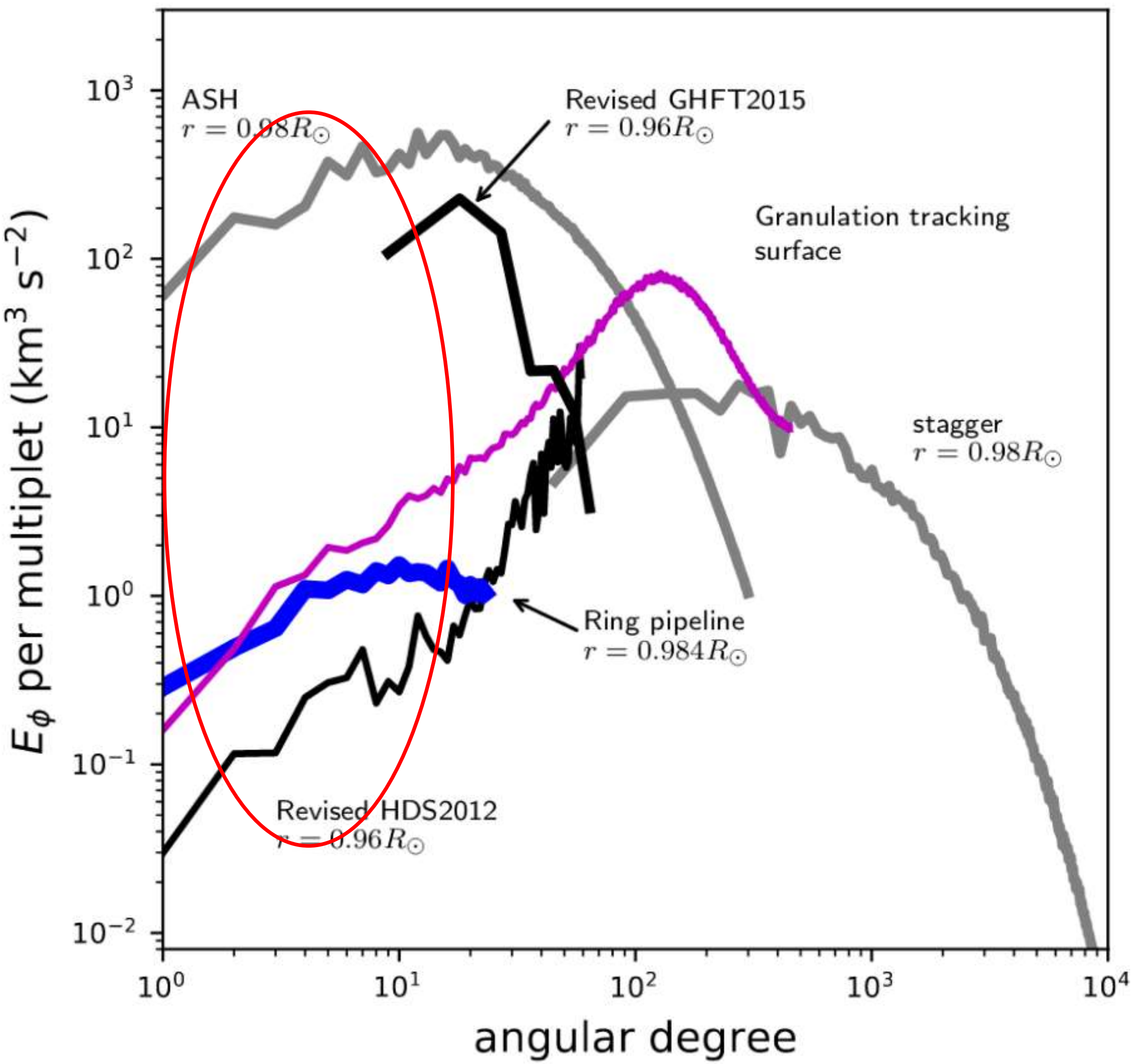
(cf next lecture)



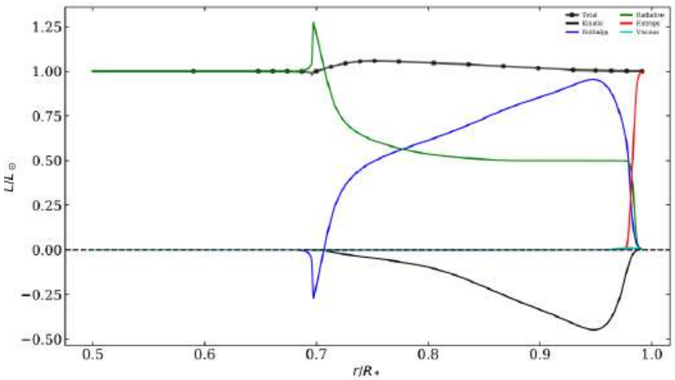
Convective Connundrum

- Current mismatch between **global models** and **helioseismic inversion** regarding giant convective cells

- **Overestimation of the Rossby number** in some global models,establishing **anti-solar** DR in turbulent solar models

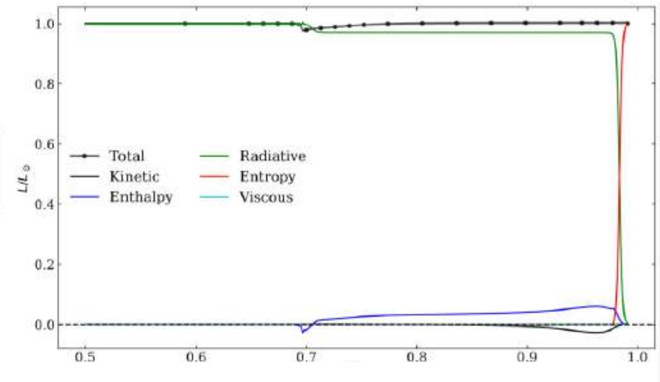


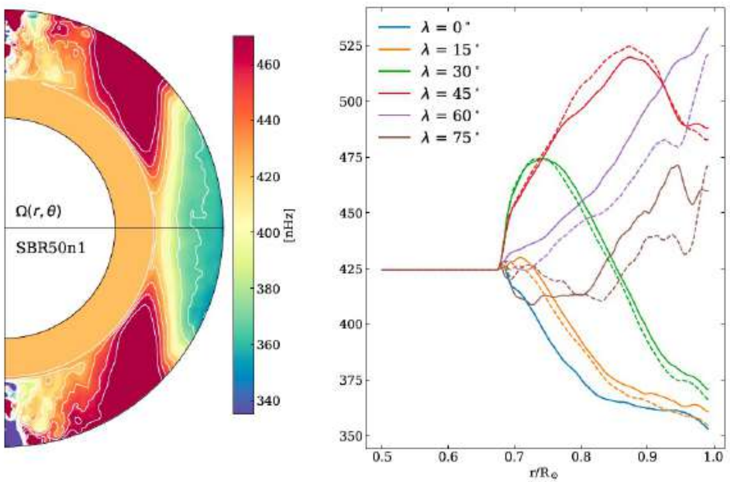
Higher Reynolds Number cases

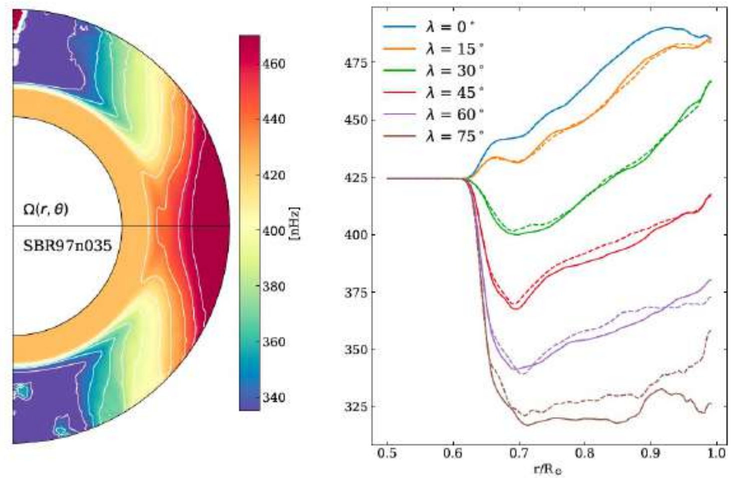


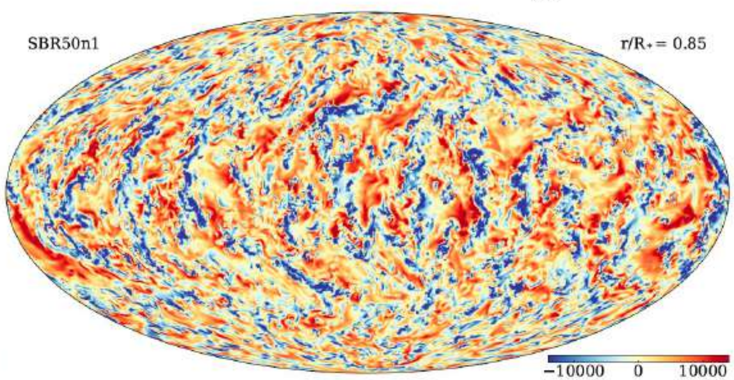
Noraz+22 (PhD)

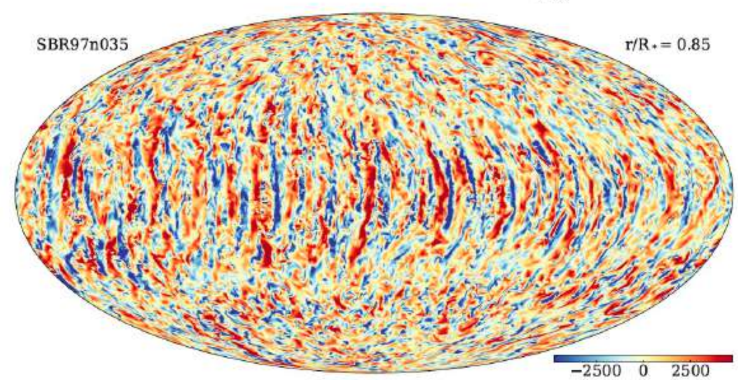
860	Re	811
4	Ro	1.49
2.45	Nu	1.04





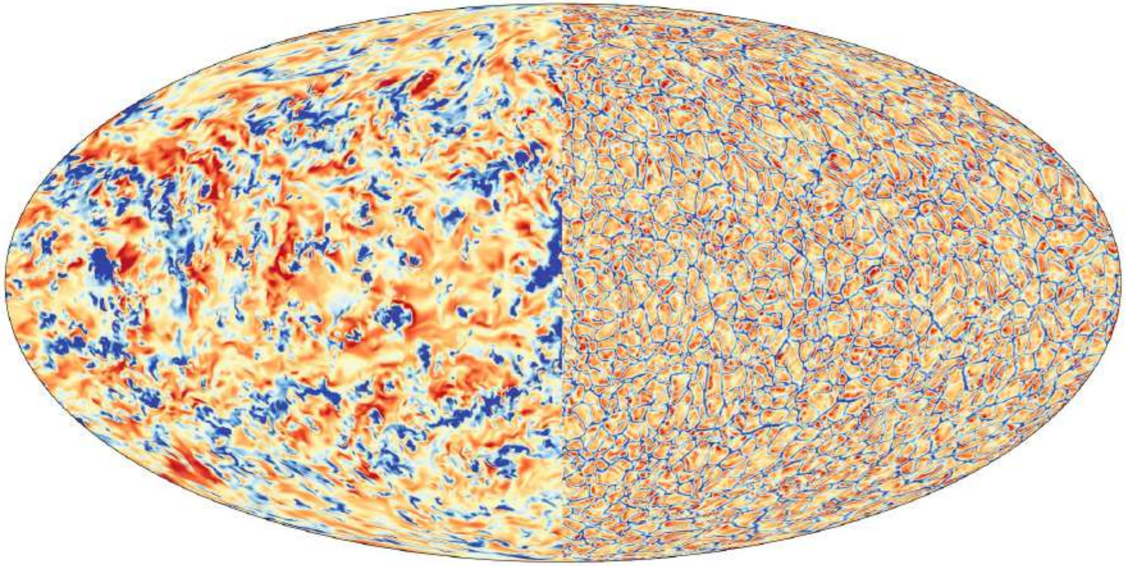


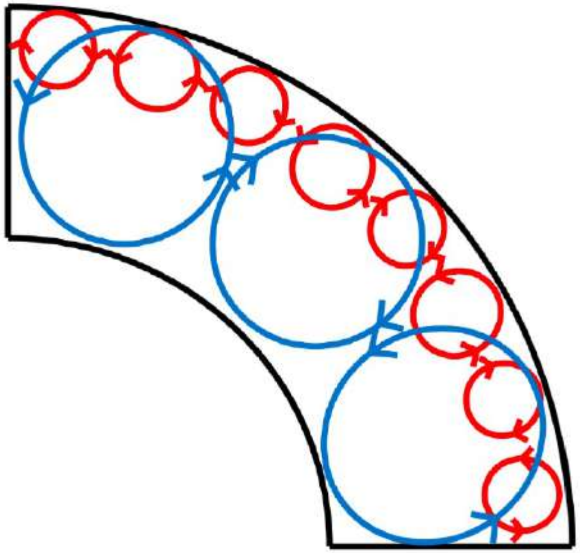




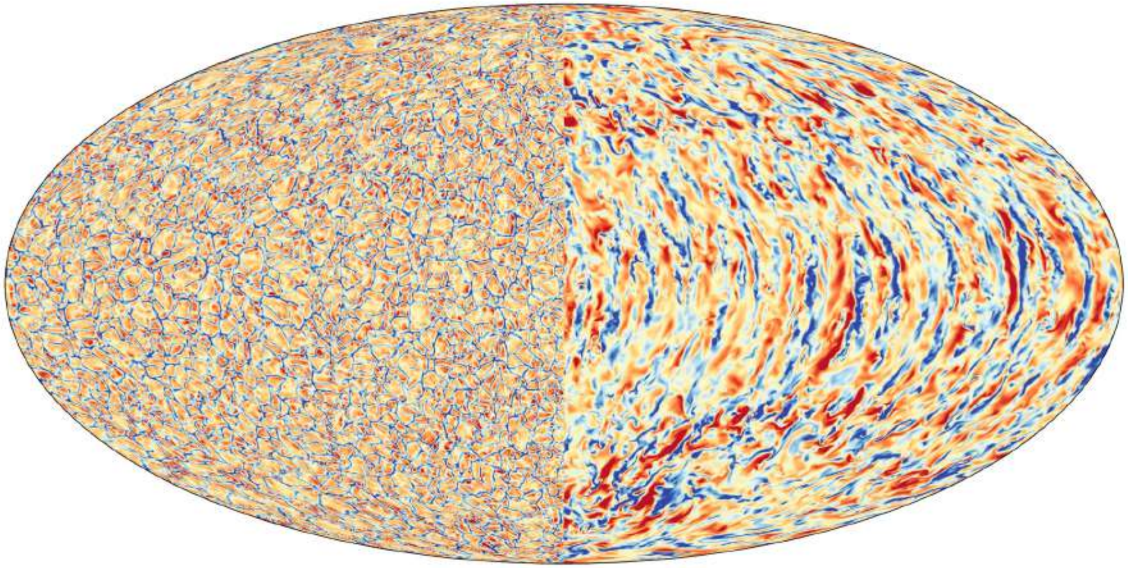
Noraz 2022 - Ph

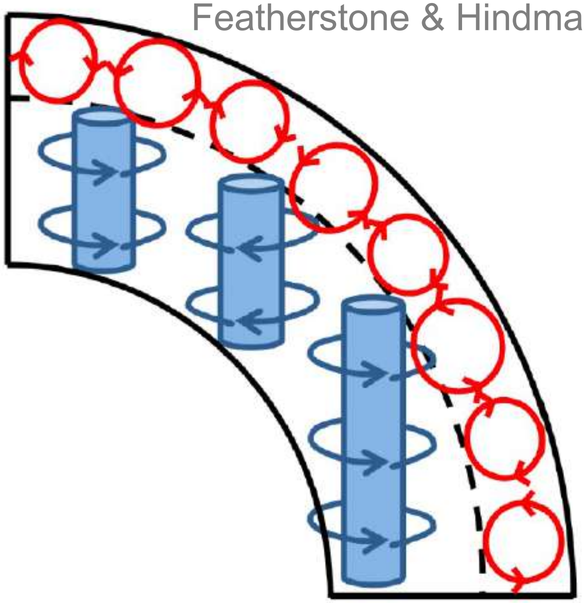
Deep Convection Dynamics





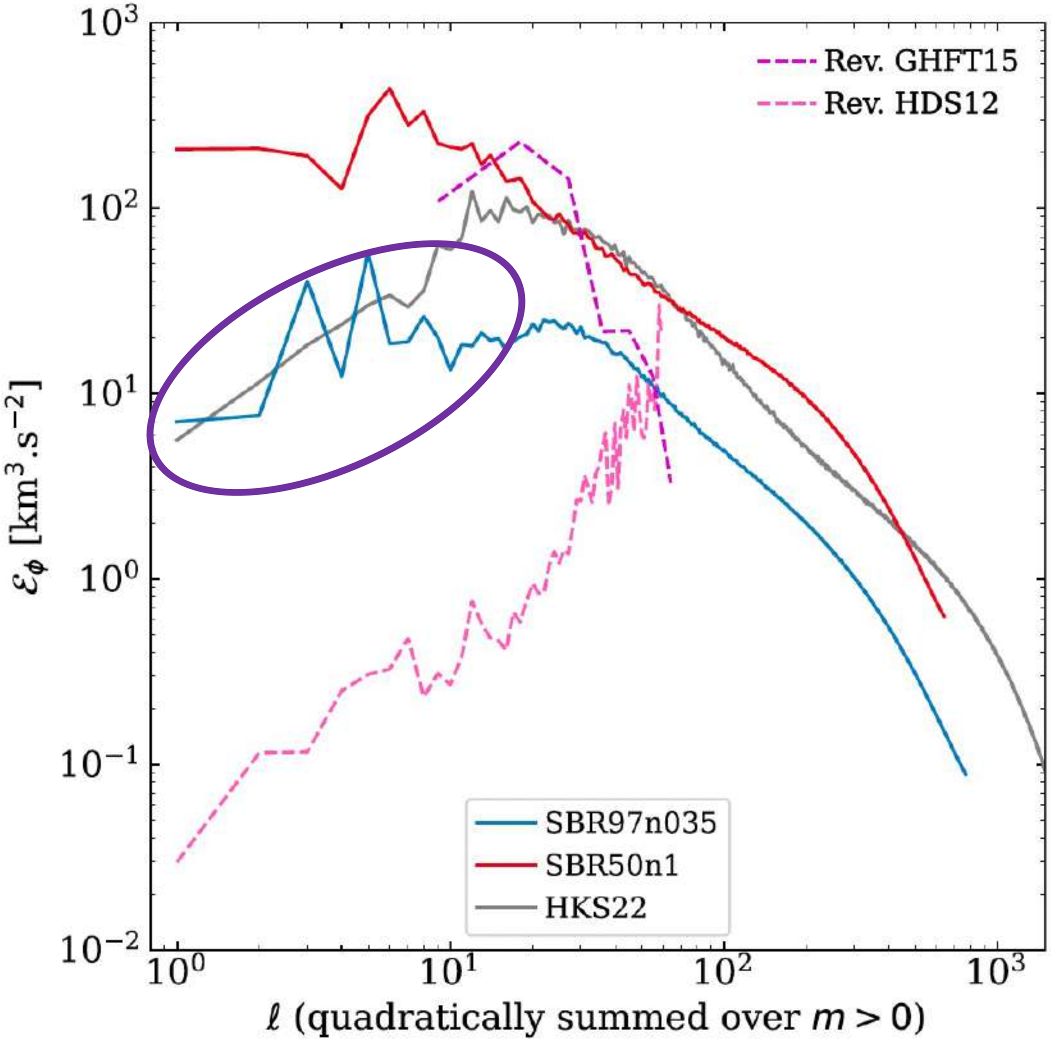
Featherstone & Hindman 2016





Noraz et al. 2025

Convective Connundrum



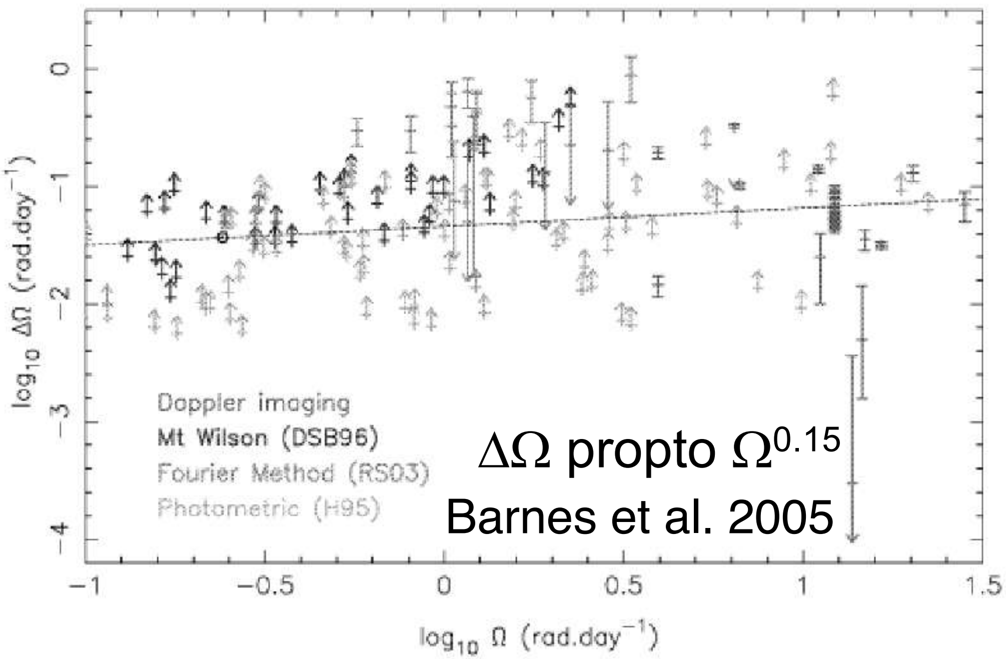
Noraz, Brun Strugarek 2005

Hotta-san
is proposing an
alternative solution
based on small scale
dynamo feedback
but as of today
lack large-scale
cyclic dynamo field

Hotta et al. 2021, 2023, 2025

Trends in Differential Rotation with Ω & Mass (Teff)

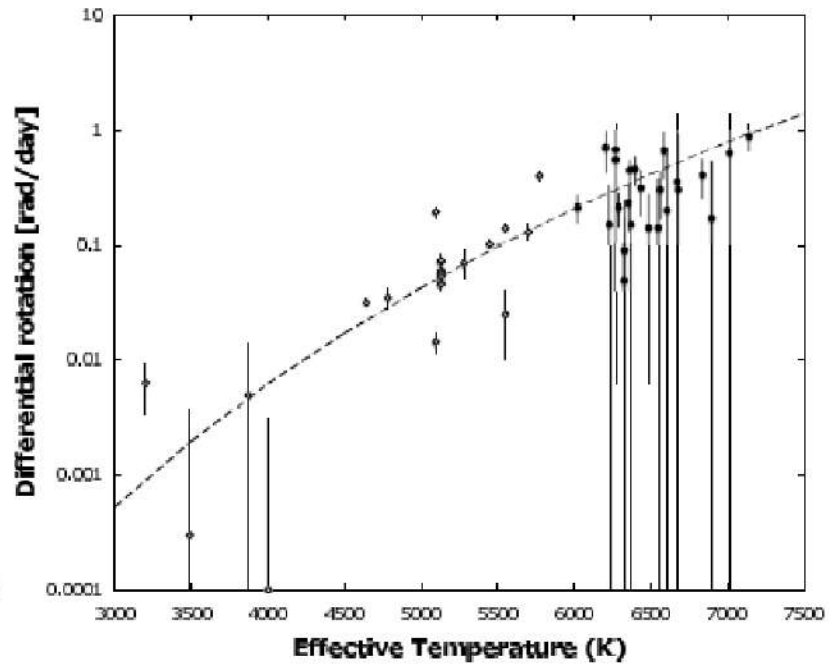
Weak trend with Ω



$\Delta\Omega$ propto $\Omega^{0.15}$

Barnes et al. 2005

$\Delta\Omega$ increases with M_*



Collier-Cameron 2007

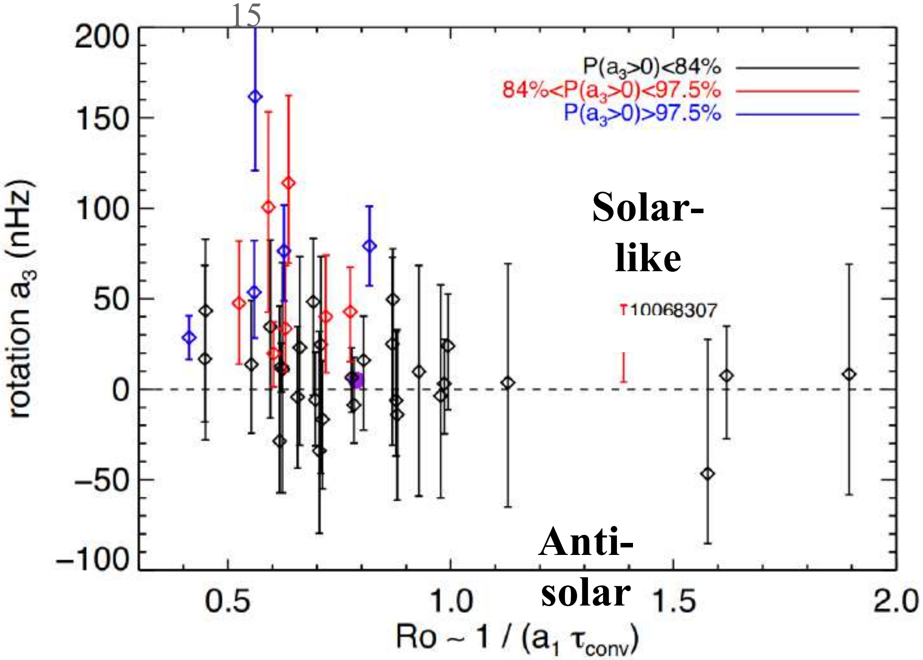
In Donahue et al. 1996: $\Delta\Omega$ propto $\Omega^{0.7}$

So currently exponent n in $\Delta\Omega$ propto Ω^{n} ranges [0.15, 0.7]

Confirming these observational scaling is key

Solar-like star Differential Rotation

Benomar+ 17;
see also Reinhold & Gizon 15

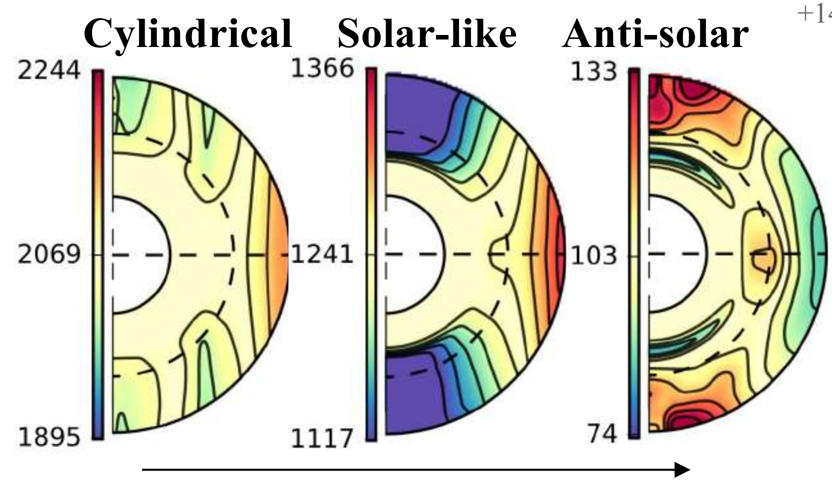


fast rotators slow rotators

- Precision decreases for slow rotators

- Different DR profiles are found in numerical simulations:

Brun+ 17 ;
see also Gilman 75,77; Kapyla+ 11; Guerrero+ 13, Gastine +14



Ω_* decreases

Rossby increases

$$Ro \sim \frac{\text{Advection}}{\text{Coriolis}}$$

Note: Anti-solar can also be understood as retrograde equator

Back of the Envelope Rossby number

$$v = c_1 \left(\frac{L_*}{\rho_{bcz} R_*^2} \right)^{1/3}$$

MLT Convective velocity

$L_* \sim M_*^4$, $R_* \sim M_*^{0.9}$, rho $\sim M_*^n$ => $v \sim M_*^{(2.2-n)/3}$

From CESAM 1-D GK star models:

$L_* \sim M_*^{4.6}$, $R_* \sim M_*^{1.3}$, rho $\sim M_*^{-6.9}$ => $v \sim M_*^3$

$$\text{Rossby number } R_{of} = v/2\Omega_* R_* = c_1 M_*^{1.7}/\Omega_*$$

The Astrophysical Journal, 836:192 (28pp), 2017 February 20

https://doi.org/10.3847/1538-4357/aa5c40

On Differential Rotation and Overshooting in Solar-like Stars

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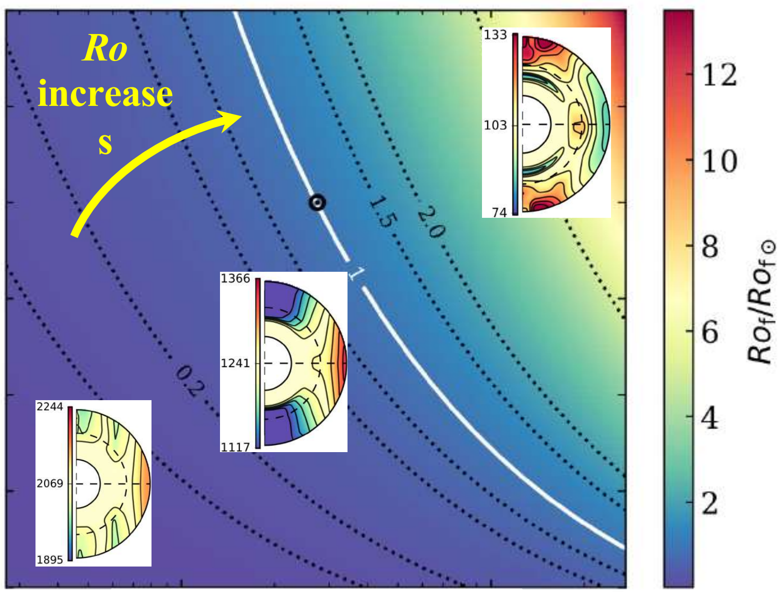
Received 2016 September 16; revised 2017 January 23; accepted 2017 January 23; published 2017 February 21

Solar-like star convection and angular velocity (rotation)

M_* increases



Ω_* decreases

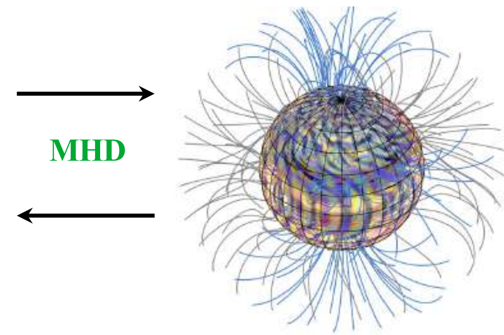
fast rotators

slow rotators

Numerical setup:

Brun, Strugarek, Noraz+ 22

- Code : ASH
- Dimensions : 3 (r,θ,φ)
- Regime : Dynamic
- → convection is explicitly resolved
- → magnetic retroaction on the flow



15 models of solar-type ~ 40M hCPU

- from 0.25 $\Omega_\odot$ to 5 $\Omega_\odot$
- from 0.5 $M_\odot$ to 1.1 $M_\odot$
- Resolution 769 x 256 x 512

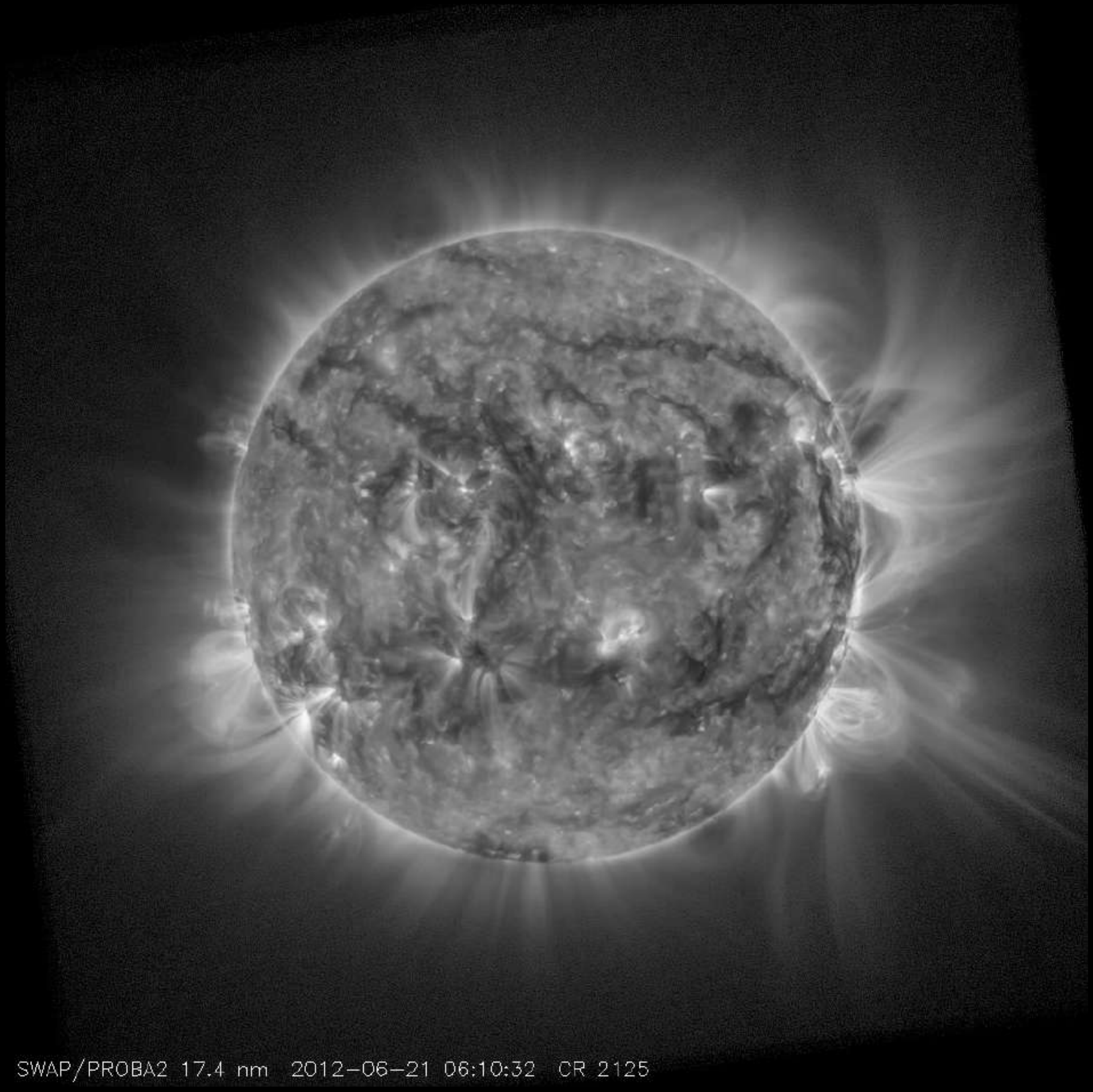
Lecture plan

I. Energy transport in stars

II. Stellar fluid dynamics
- From particle description to fluid description in plasmas
- Turbulence: basic concepts
- Convection simulation: some exemples

III. Dynamo effect in turbulent convective zones of stars
- The Sun's magnetic field
- The dynamo effect: fundamental ingredients
- The dynamo effect: kinematics vs. Dynamics
- Stellar magnetism and dynamo

What do we know about solar magnetism?



Maunder minimum (~1650-1715)

The Sun has been magnetically active for very long time

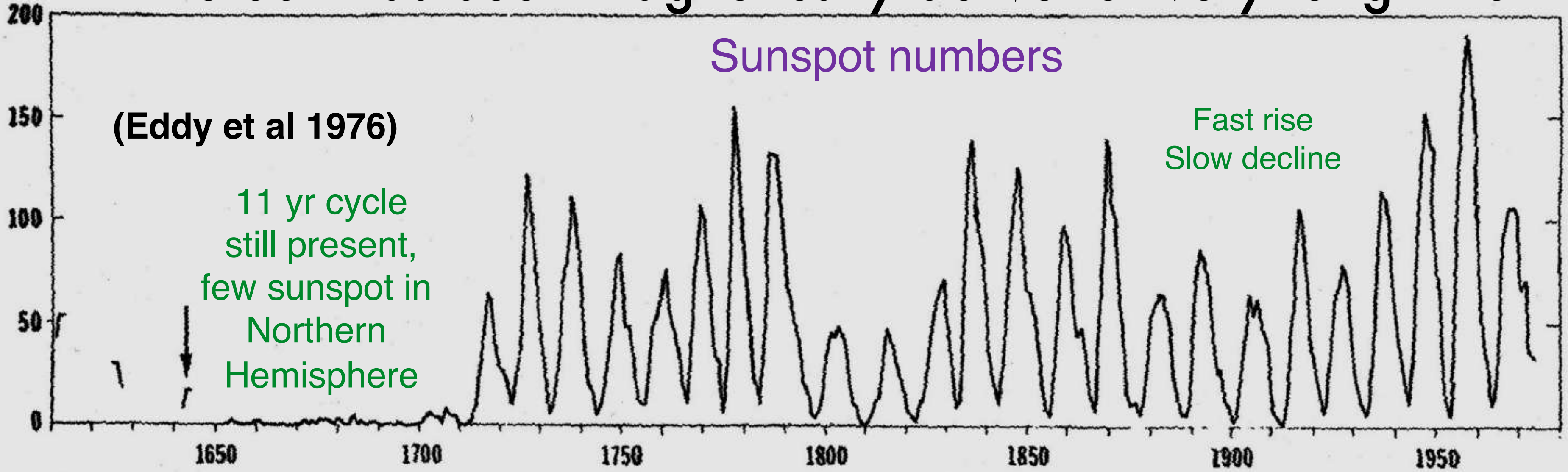
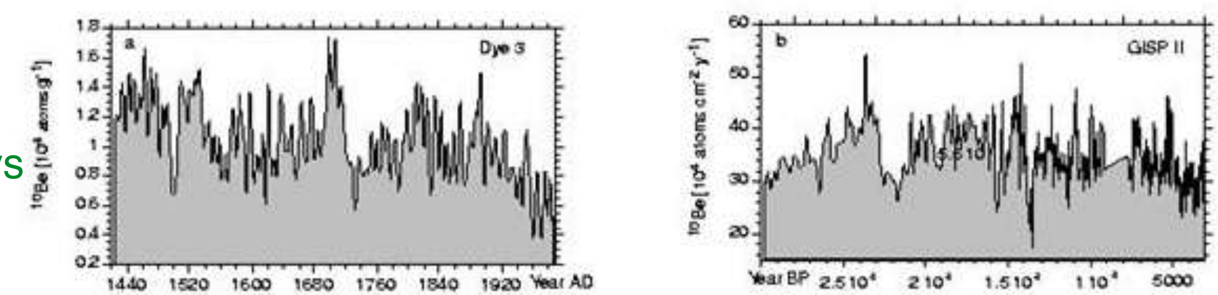


Fig. 1. Annual mean sunspot numbers, A.D. 1610–1974, from Waldmeier (1961) and Eddy (1976). Arrow marks the period of this study, 1642–1644.

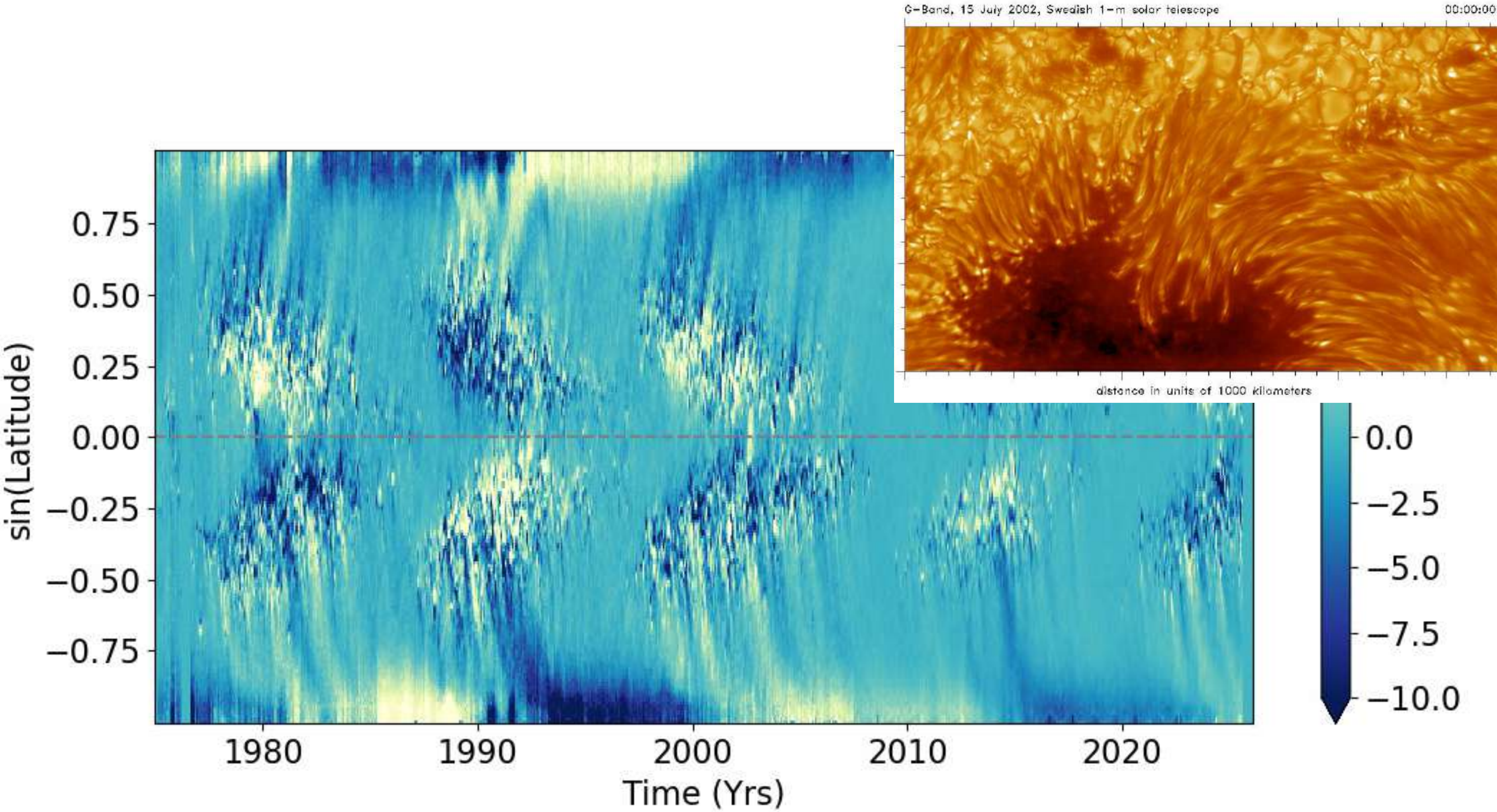
Anti
correlation
With cosmic rays



BEER (2000)

Other methods: Be10 (ice cores over 2 yr) or C14 (tree rings over 30 yr)
Presence of other minima (frequency ~200 years), e.g., Sporer (1420-1530)
Modulation of the cycle of ~100 years (Gleissberg cycle)

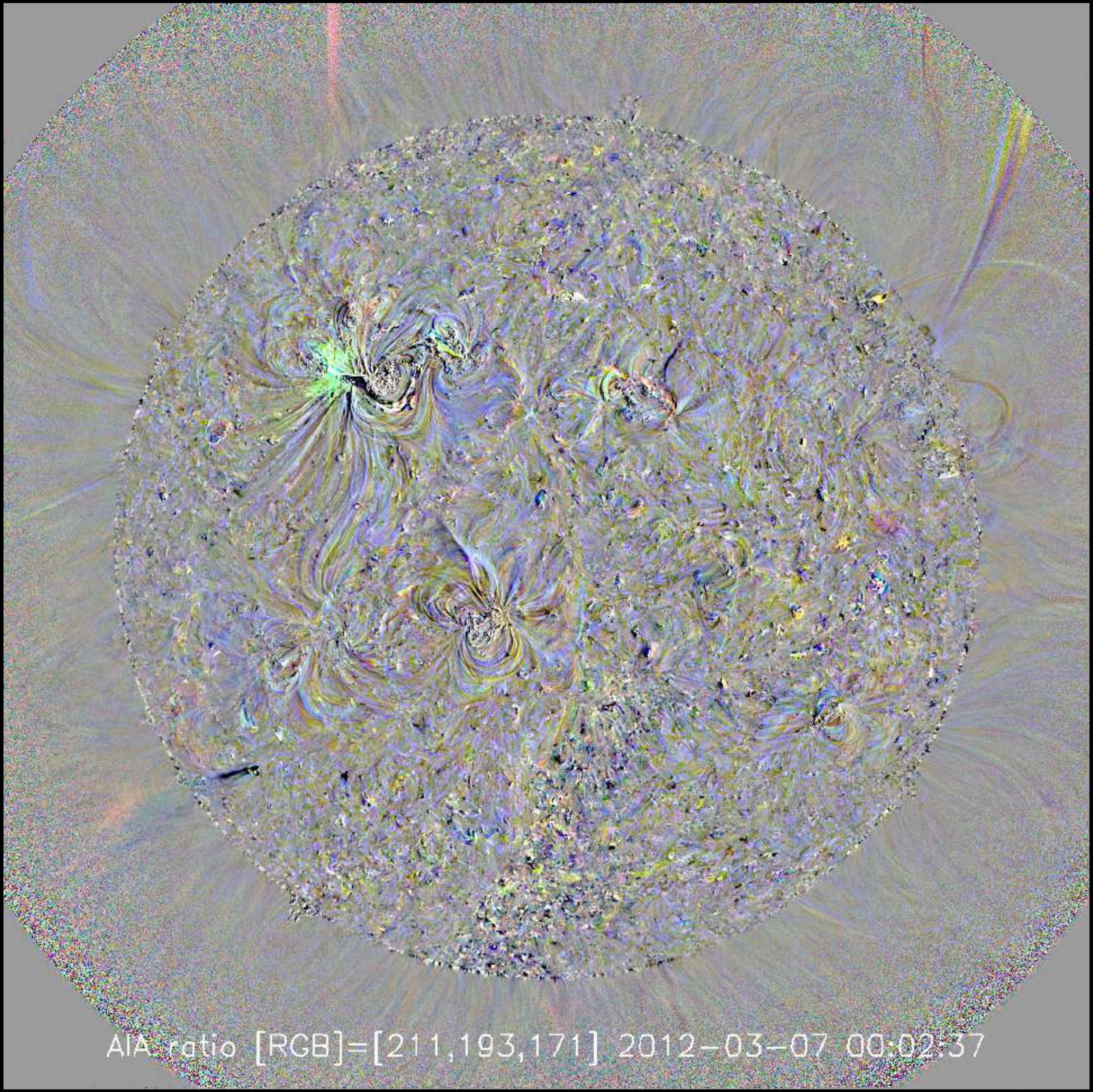
What do we know about solar magnetism?



Butterfly diagram: sunspot migration towards equator during 11 yr cycle

What do we know about solar magnetism?

cf. Thursday lecture



Lecture plan

I. Energy transport in stars

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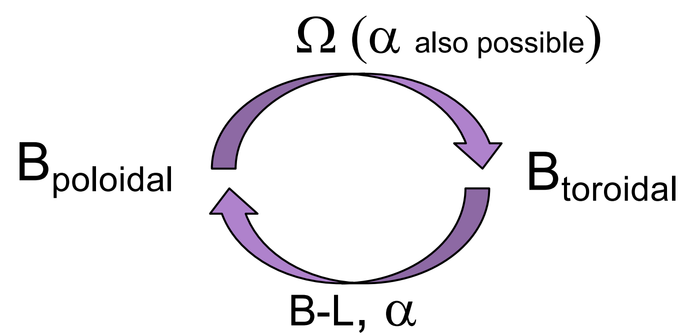
- The dynamo effect: kinematics vs. dynamics

Magnétisme dans le Soleil (et les étoiles): l'effet Dynamo

Dynamo effect: It is the property of a conductive fluid to generate a magnetic field through its motions (by self-induction) and to maintain this field against Ohmic dissipation.

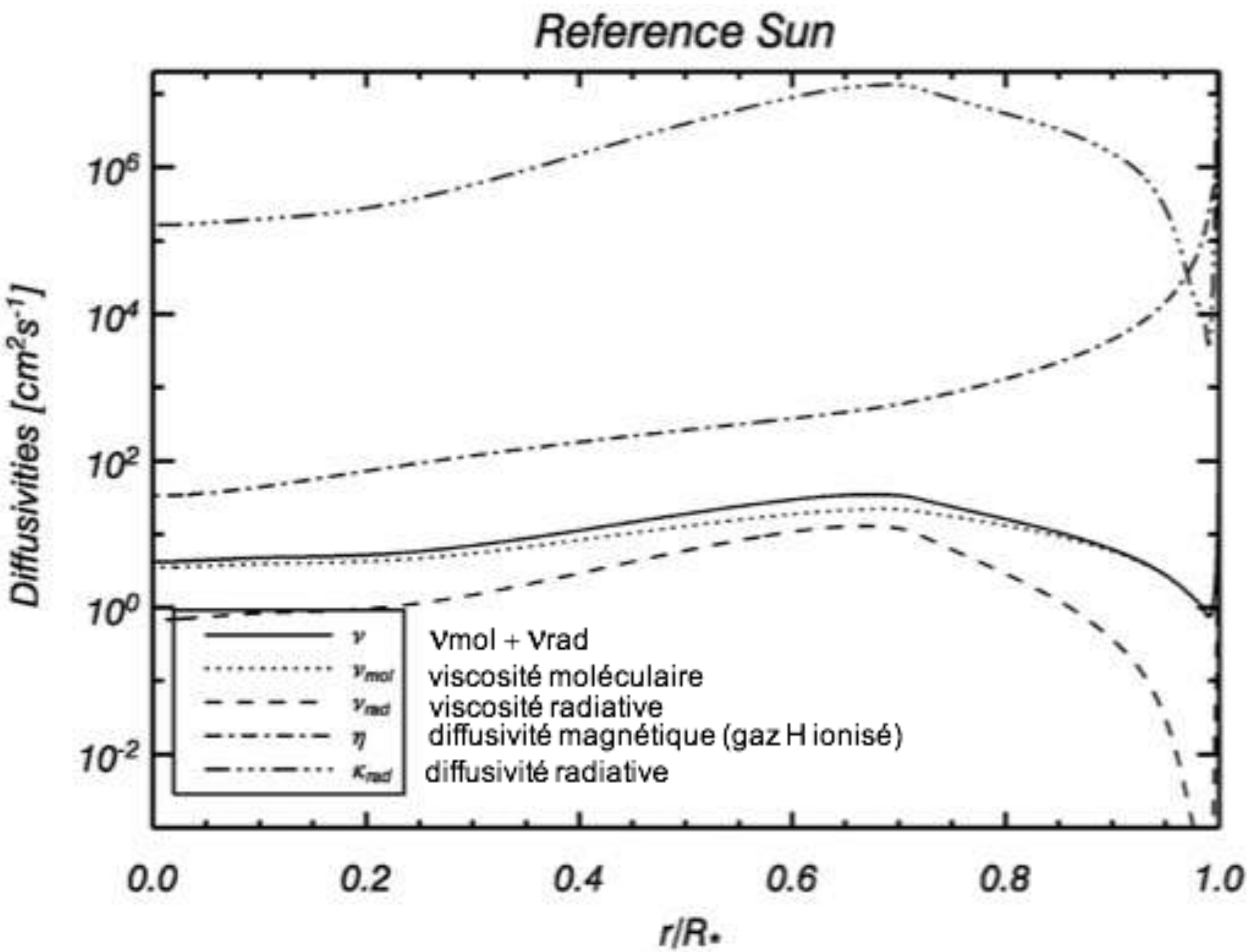
It is fundamentally a tri dimensional process, there is for instance an anti-dynamo theorem (Cowling) forbidding a purely axisymetric dynamo

Dynamo loop:



It is key that both components (poloidal and toroidal) of the magnetic field be regenerated

Atomic diffusivities in the Sun (pure H gas)



$$\nu_{rad} = \frac{4}{15}\frac{a}{c}\frac{T^4}{\rho^2 \kappa_{opa}}$$

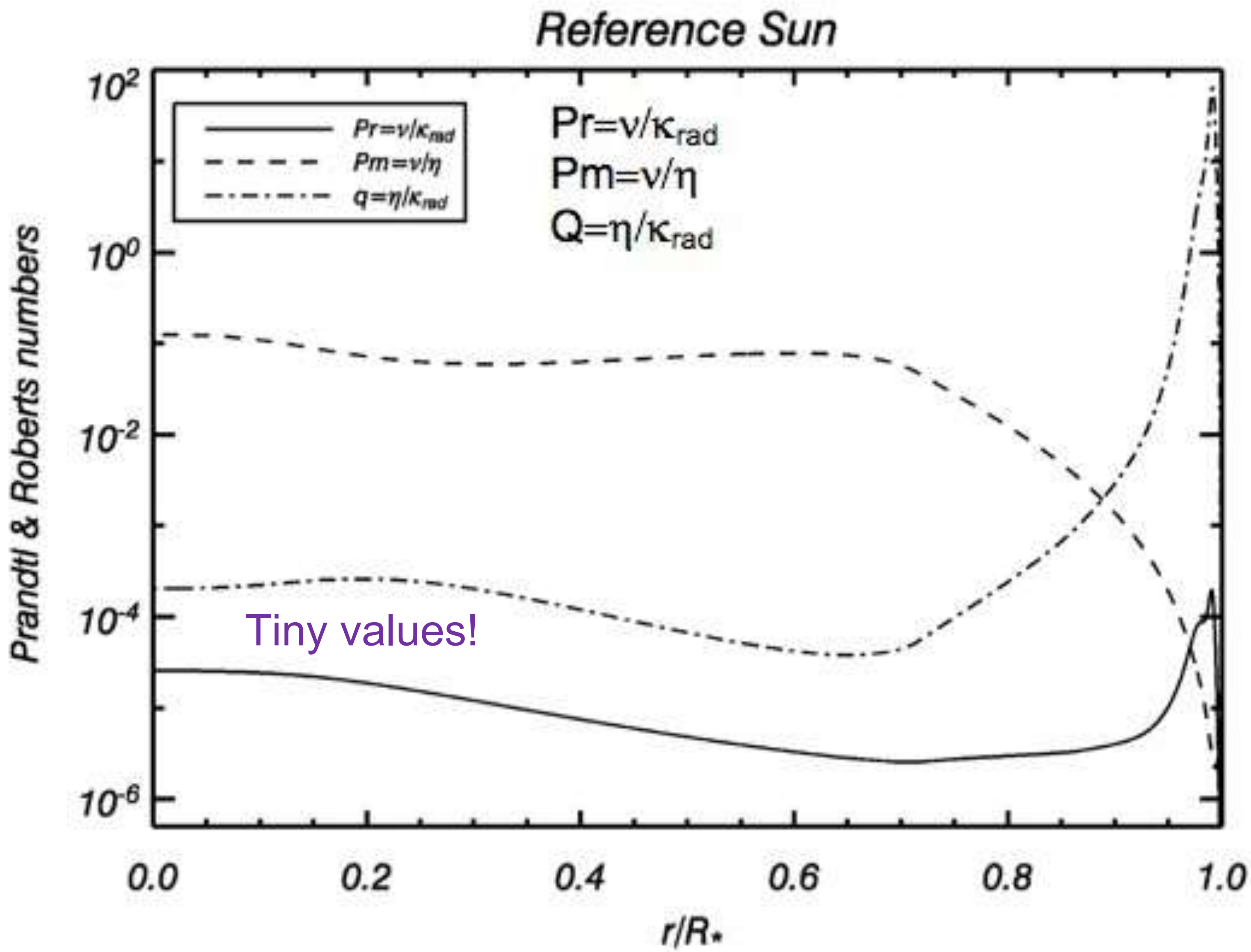
$$\nu_{mol} = \frac{2.2\,10^{-15}}{\ln\Lambda}\frac{T^{5/2}}{\rho}$$

$$\eta = 5.2\,10^{11} \ln\Lambda\, T^{-3/2}$$

$$\kappa_{rad} = \frac{\chi}{\rho c_p} = \frac{4}{3}\frac{acT^3}{\rho^2 c_p \kappa_{opa}}$$

See Zeldovich et al. 1983, for a discussion of the formula of ν, η, κ

Prandtl numbers inside the Sun



Maxwell equations (cgs)

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (4)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad (5)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \quad (7)$$

Note: 3 types of magnetic materials ($B=\mu_m H$, B magnetic field):
Diamagnetism (magnetic permeability $\mu_m<1$): most materials are diamagnetic (e.g., water) (repulsion limiting the imposed external field) (full electron shells)
Paramagnetic ($\mu_m > 1$): weak attraction (non-full electron shells) (e.g., aluminum)
Ferromagnetic ($\mu_m >> 1$): strong attraction, existence of magnetic domains due to favorable orientation of electron spins, residual magnetization (hysteresis) (e.g., iron).

Back to Induction equation (I)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \Delta \mathbf{B}$$

(cas η cst)

If the plasma is at rest, the equation becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B}$$

This is a diffusion equation, B-field decreases on a sphere of radius R in an Ohmic timescale of:

$$\tau_\eta = \frac{R^2}{\pi^2 \eta}$$

For instance in a conductor used in a laboratory, τ_η is small (10 s for a copper sphere of radius 1m), but in cosmic conductors, it can be huge (> 10^{10} yr)

By contrast, if the plasma is in motion (and that its resistance is negligible) the equation becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

This means that magnetic field lines are « frozen » (Alfvén theorem) in the plasma

Back to Induction equation (II)

The magnetic Reynolds number Rm allows to asses the regime in which the system under study is at. Usually, Rm is small in laboratory experiment (Rm ~1 & < 50) and very large in cosmic bodies (> 10^6).Theoretically there is a sustained dynamo effect if Rm is sufficient large

This means that even though electric current in laboratory conductors are mainly determined by the conductivity σ, in a cosmic body, a change by say a factor 10 of σ, does not induce a significant change of B. The conductivity is here to determined the electric field E (weak) needed to the presence of such currents (Cowling 1957).

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} - \frac{\mathbf{v} \times \mathbf{B}}{c}$$

Note: the first term of the induction equation can be split into:

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{v}$$

a term (1st) of distorsion and stretching of B, a term of advective transport, and the last term is linked to compressibility of the plasma (null if Div. V=0)

Mean field theory (I)

On commence avec l'équation d'induction

$$\partial_t \mathbf{B} = \mathbf{\nabla} \times (\mathbf{U} \times \mathbf{B}) + \eta \Delta \mathbf{B}$$

Supposons que l'on peut séparer les champs **B** et **u** en grandes et petites échelles, telles que

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b} \qquad \mathbf{U} = \mathbf{U}_0 + \mathbf{u}$$

où $\mathbf{B}_0$ varie sur un échelle typique L et **b** sur une échelle l, avec l<<L

On définit alors l'opérateur de moyenne aux grandes échelles tel que

$$\langle \mathbf{B} \rangle = \mathbf{B}_0$$

(bien sur <u> = <b> =0)

Mean field theory (II)

Alors, si on essaye d'écrire l'équation régissant **B**0, on obtient

$$\partial_t \mathbf{B}_0 = \nabla \times (\mathcal{E} + \mathbf{U}_0 \times \mathbf{B}_0) + \eta \Delta \mathbf{B}_0$$

où la force électro-motrice s'écrit $\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$

Et l'équation sur les fluctuations devient

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{B}_0 + \mathbf{U}_0 \times \mathbf{b}) + \nabla \times \mathbf{G} + \eta \Delta \mathbf{b}$$

où $\mathbf{G} = \mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle$ est un terme 'pain in the neck'....

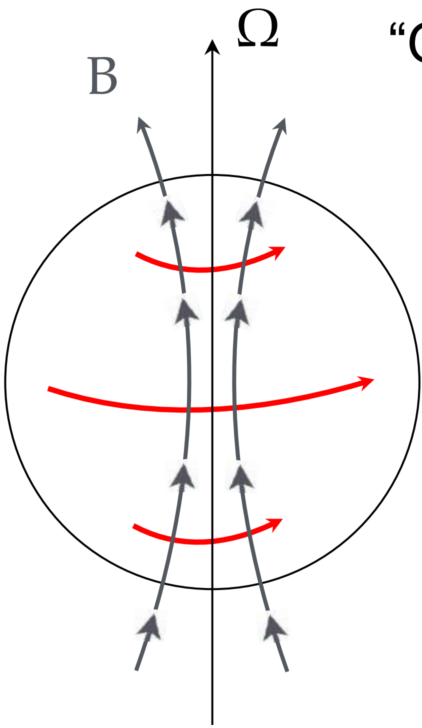
Si **G** est petit, alors **b** dépend uniquement de **B**0 et **u**, et on peut ainsi développer $\mathcal{E}$ autour de **B**0 :

$$\mathcal{E}_i = \alpha_{ij} \langle B_j \rangle + \beta_{ijk} \partial_{x_k} \langle B_j \rangle + \ldots$$

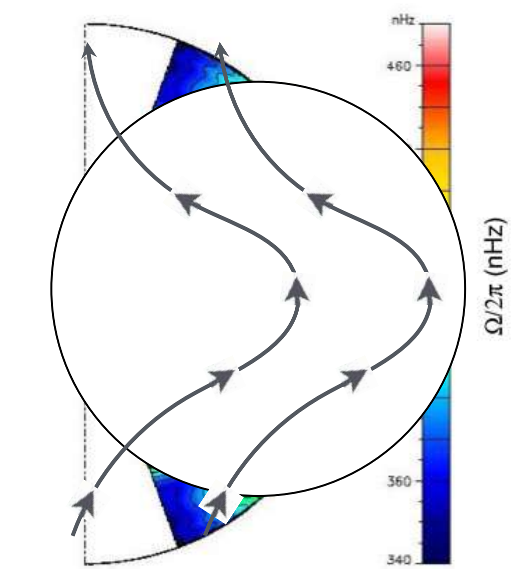
Mean field theory (III)

We then finally obtain a single equation for the large-scale field B_0:

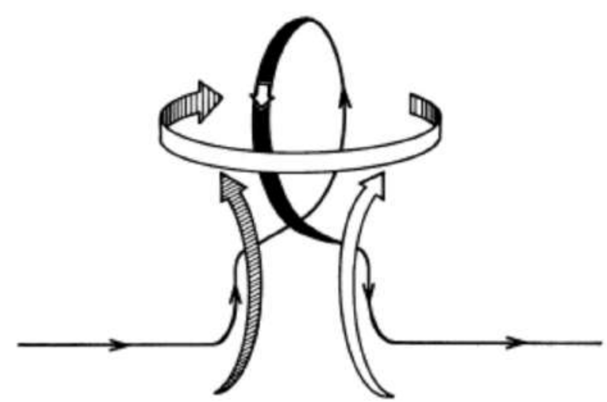
$$\partial_t \mathbf{B}_0 = \boldsymbol{\nabla} \times (\alpha \mathbf{B}_0 + \mathbf{U}_0 \times \mathbf{B}_0) + (\eta + \beta)\, \Delta \mathbf{B}_0$$



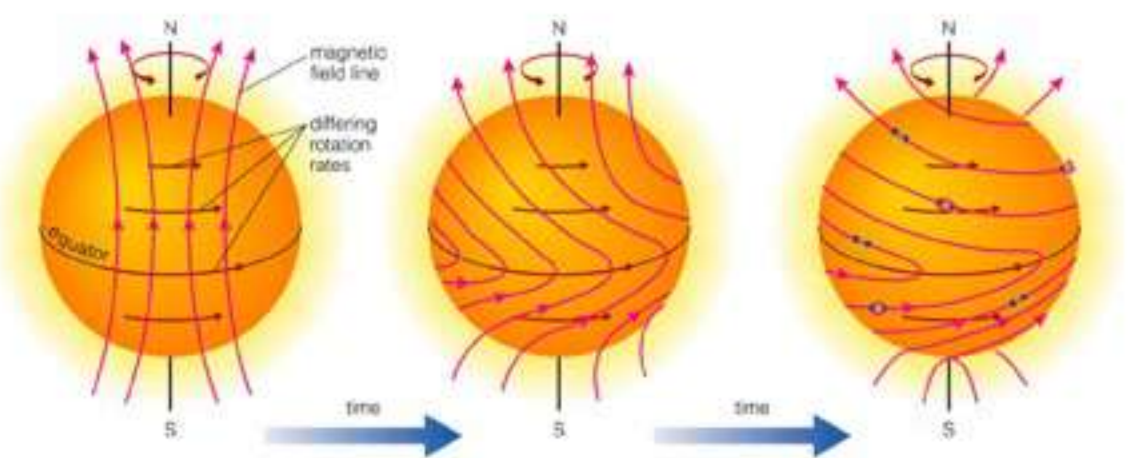
"Omega" Effect

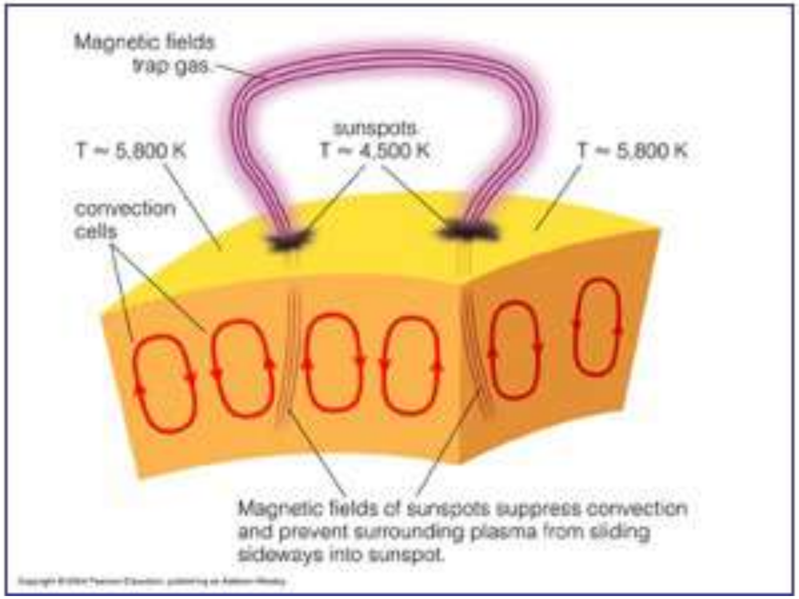


"alpha" effect



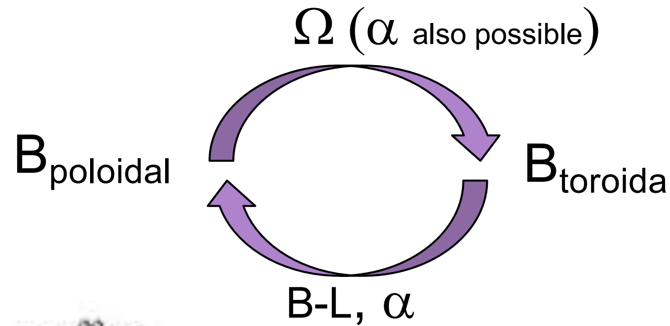
Transport and generation of toroidal magnetic field





Dynamos α-Ω vs Babcock-Leighton

Dynamo loop:



Plusieurs types de dynamo:

$\alpha\!-\!\Omega$, α^2, $\alpha^2\,\Omega$, BL- Ω

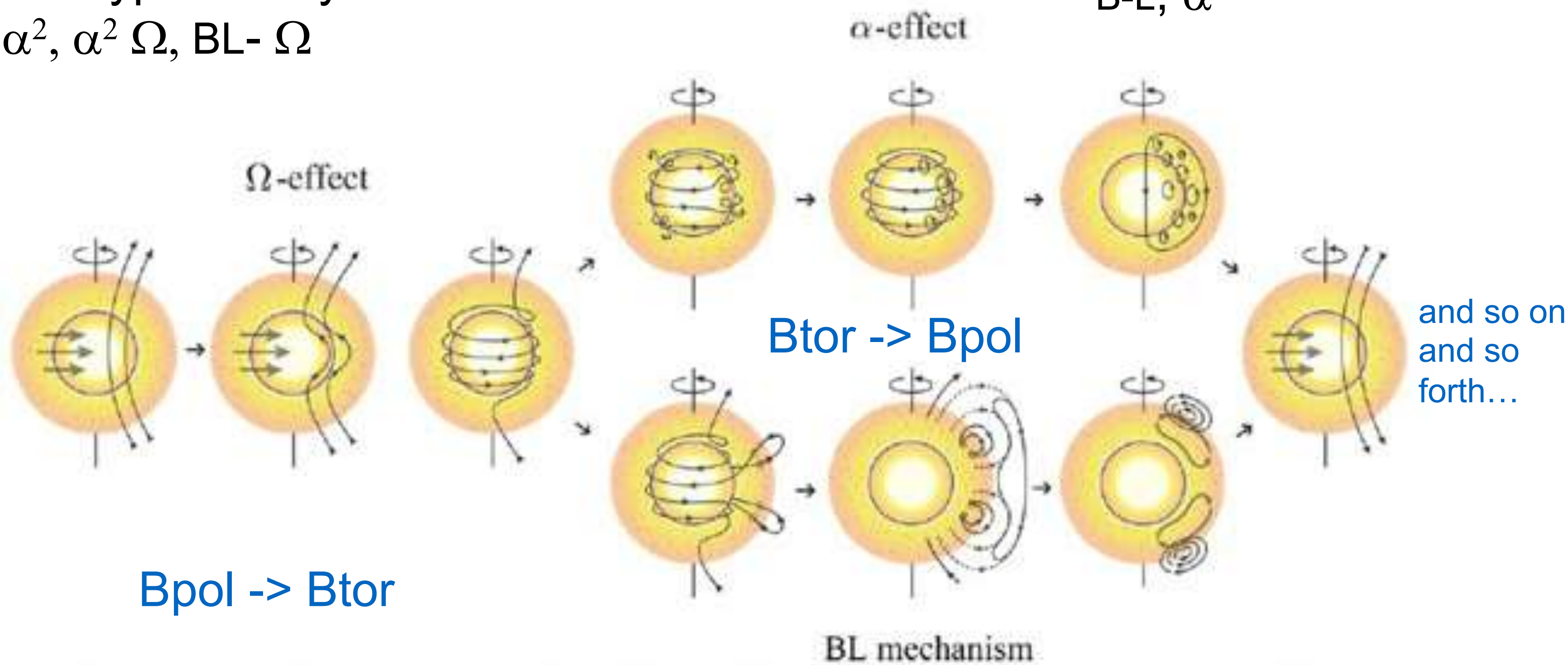
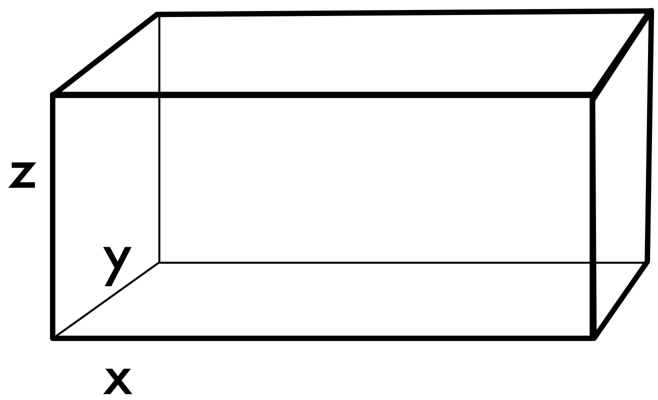


Figure 1. Sketch of the main processes at work in our solar dynamo model. The Ω-effect (left) depicts the transformation of a primary poloidal field into a toroidal field by means of the differential rotation. The poloidal field regeneration is next accomplished either by the α-effect (top) and/or by the Babcock-Leighton mechanism (bottom). In the α-effect case, the toroidal field at the base of the convection zone is subject to cyclonic turbulence. Secondary small-scale poloidal fields are thereby created, and produce on average a new, large-scale, poloidal field. In the Babcock-Leighton mechanism, the primary process for poloidal field regeneration is the formation of sunspots at the solar surface from the rise of buoyant toroidal magnetic flux tubes from the base of the convection zone. The magnetic fields of those sunspots nearest to the equator in each hemisphere diffuse and reconnect, while the field due to those sunspots closer to the poles has a polarity opposite to the current one, which initiates a polarity reversal. The newly formed polar magnetic flux is transported by the meridional flow to the deeper layers of the convection zone, thereby creating a new large-scale poloidal field.

Cartesian dynamo example



$$\mathbf{U} = \Omega_0 z \mathbf{e}_y$$

$$\mathbf{B} = \boldsymbol{\nabla} \times (A\mathbf{e}_y) + B_t \mathbf{e}_y$$

poloidal toroidal

One gets that a dynamo field is maintained if (see derivation): $k > \left(\frac{\alpha\Omega_0}{2\eta^2}\right)^{1/3}$

In that case, $B_t = B_0 \exp^{i(\omega t + kx)}$

with $\omega = \sqrt{-\alpha\Omega_0 k/2} \cdot (1+i) - \eta k^2 i$

Cartesian dynamo derivation (I)

a) To derive the equation for A and B, we start by injecting $\mathbf{B} = \nabla \times A\mathbf{e}_y + B\mathbf{e}_y$ into the mean field induction equation:

$$\frac{\partial(\nabla \times A\mathbf{e}_y + B\mathbf{e}_y)}{\partial t} = \nabla \times (\langle \mathbf{V} \rangle \times (\nabla \times A\mathbf{e}_y + B\mathbf{e}_y) + \alpha(\nabla \times A\mathbf{e}_y + B\mathbf{e}_y) - \eta_t \nabla \times (\nabla \times A\mathbf{e}_y + B\mathbf{e}_y)),$$

We notice that for the equation for A, we have a $\nabla\times$ operator that can be simplified by uncurling the equation, since for instance the time derivative and the *curl* operator can be swapped. We can then project along $\mathbf{e}_y$ both the uncurled induction equation and the induction equation and obtain an equation for A and B respectively. Recall that for the equation for B we neglect the α-effect and only keep the ω-effect. Also recall that $\langle \mathbf{V} \rangle = (0, \omega_0 z, 0)$ and the mean magnetic field $\mathbf{B}$ depends only on x and t. With all these assumptions, we find the following set of equations for A and B:

$$\frac{\partial A}{\partial t} = \alpha B + \eta_t \frac{\partial^2 A}{\partial x^2}$$

$$\frac{\partial B}{\partial t} = \omega_0 \frac{\partial A}{\partial x} + \eta_t \frac{\partial^2 B}{\partial x^2}$$

Cartesian dynamo derivation (II)

b) Introducing the solutions $(A, B) = (A_0, B_0)e^{i(kx+\sigma t)}$ into the above equations, results in the following dispersion relation:

$$(i\sigma + \eta_t k^2)^2 = \alpha_0\omega_0 ik$$

c) Expanding the relation above, one gets:

$$\sigma^2 - 2i\sigma\eta_t k^2 - \eta_t^2 k^4 + \alpha_0\omega_0 ik = 0.$$

Since $\sigma = \sigma_r + i\sigma_i$ is a complex number, one furthers gets:

$$\sigma_r^2 + 2i\sigma_r\sigma_i - \sigma_i^2 - 2i\sigma_r\eta_t k^2 + 2\sigma_i\eta_t k^2 - \eta_t^2 k^4 + \alpha_0\omega_0 ik = 0.$$

Separating the imaginary from the real parts we obtain the following 2 equations:

$$\begin{aligned} \sigma_r^2 &- \sigma_i^2 + 2\sigma_i\eta_t k^2 - \eta_t^2 k^4 = 0 \\ 2i\sigma_r\sigma_i &- 2i\sigma_r\eta_t k^2 + \alpha_0\omega_0 ik = 0 \end{aligned}$$

The first equation yields a quadratic equation for σ_i: The determinant is $\Delta = 4\sigma_r^2$, and $\sigma_i = \eta_t k^2 \pm |\sigma_r|$. From the second equation one deduces that: $2\sigma_r(\sigma_i - \eta_t k^2) = -\alpha_0\omega_0 k$. We select solutions such that $\alpha_0\omega_0 < 0$, and after injecting the expression of σ_i, one finds that only $\sigma_r^2 = -\alpha_0\omega_0 k/2$ can be real (recall that σ_r is real by definition).

Cartesian dynamo derivation (III)

The solution is $\sigma = i\eta_t k^2 + (1 \pm i)\sqrt{|\alpha_0\omega_0 k/2|}$. The threshold for dynamo action is given by $-\sigma_i = -\eta_t k^2 \mp \sqrt{|\alpha_0\omega_0 k/2|} \geq 0$, so this leads to selecting the solution $-\eta_t k^2 + \sqrt{|\alpha_0\omega_0 k/2|}$, hence the marginal state $\sigma_i = 0$, implies that

$$\eta_t k^2 = \sqrt{|\alpha_0\omega_0 k/2|}.$$

Hence the product $\alpha_0\omega_0$ determines the growth rate against Ohmic dissipation η_t.
The frequency of oscillation of the dynamo wave is given by the $\sigma_r = \pm\sqrt{|\alpha_0\omega_0 k/2|}$. We choose the negative solution $\sigma_r = -\sqrt{|\alpha_0\omega_0 k/2|}$, to have a wave propagating in the positive x direction, such as to get $\exp(\sigma_r t + kx)$, i.e. a direction of propagation corresponding to an equatorward dynamo branch as in the Sun's low latitude activity branch. Positive solution is also correct, but the main dynamo branch is then poleward unlike the Sun's.

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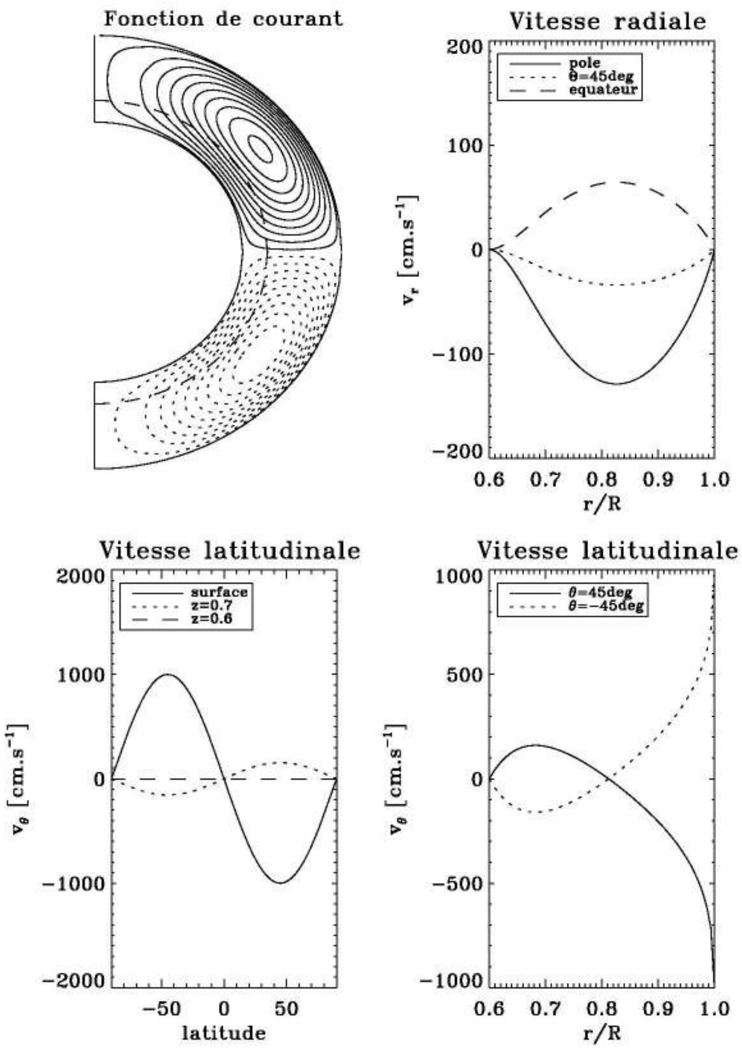
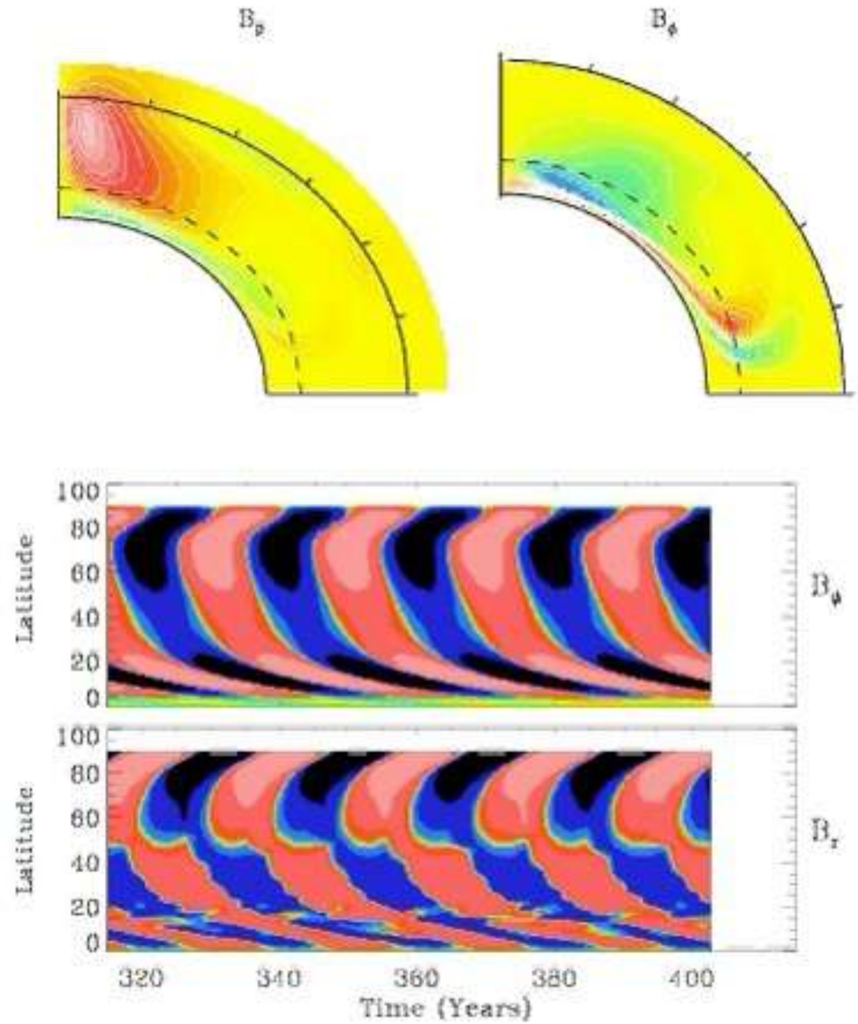
2D Mean Field models: Babcock-Leighton

1 single cell per hemisphere





Jouve & Brun, 2007 A&A

alpha-omega dynamo: Parker-Yoshimura rule

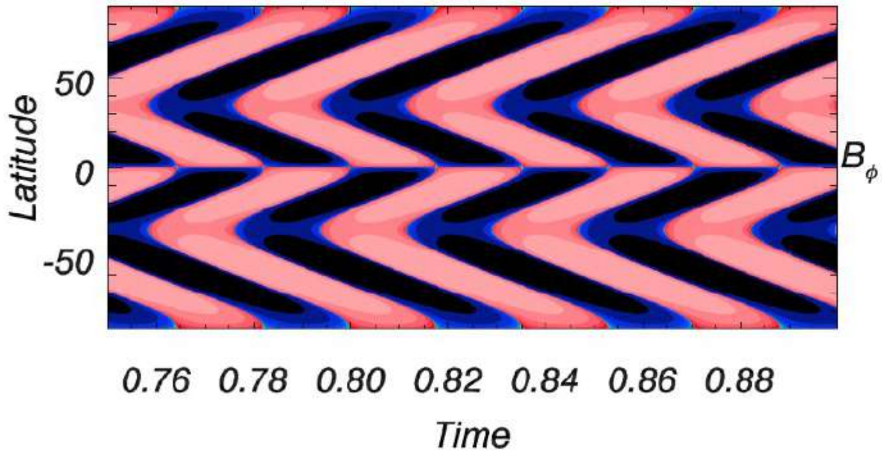
Currently, in order to explain how the solar large-scale dynamo operates while integrating constraints from helioseismology, kinematic models of the solar dynamo are based on the so-called interface dynamo (Parker, 1993). Here, the tachocline plays a dominant role by locating the ω-effect at the base of the convective zone, where the toroidal field can be effectively amplified and stored over several years (solving in an elegant way the problem of the large difference between the characteristic time scales of the convection and that of the cycle). However, there are two different approaches to converting the toroidal field into a poloidal field, the usual α-effect dynamo (α-ω, (Charbonneau and MacGregor, 1997) or flux transport ones using a Babcock-Leighton term at the surface (Babcock, 1961; Leighton, 1969; Choudhuri et al., 1995). In order to reproduce the solar cycle assuming the helioseismically inferred differential rotation, the alpha effect must be confined near the base of the convective envelope. Distributed alpha-effect do not work anymore. This is linked to the so-called Parker-Yoshimura rule (Parker, 1955; Yoshimura, 1975). The product of the the alpha effect with the shear of Omega dictates the propagation direction of the α-ω dynamo waves.

$$\mathbf{S} = -\varpi\alpha\hat{\mathbf{e}}_\phi \times \boldsymbol{\nabla}\frac{\Omega}{\Omega_\odot},$$

with $\varpi = r\sin\theta$. In order to have the main mid-latitude-equatorial branch of the dynamo waves mimicking the solar butterfly magnetic activity diagram (see Figure 2.8) the sign of $\mathbf{S}$ must be positive in the northern hemisphere, i.e. in the direction of $\hat{\mathbf{e}}_\theta$ (and opposite (negative) in the southern one). It hence depends on the sign of $d\Omega/dr$ and α. The sign of $d\Omega/dr$ is known thanks to helioseismic inversions, it is positive at low latitude except near the surface. Hence the sign of α in the northern hemisphere must be negative. We can have an idea of the sign of the α-effect by following (Moffatt, 1978). It can be shown that the α-effect can be related to the kinetic helicity $H_k = \mathbf{v}\cdot\omega_v$ contained in the system, e.g.: $\alpha = -\frac{\tau_c}{3}\langle\mathbf{v}\cdot\omega_v\rangle$, with $\omega_v = \boldsymbol{\nabla}\times\mathbf{v}$ the vorticity and τ_c, the convective turnover time scale.

alpha-omega dynamo: Parker-Yoshimura rule

Solar like



Anti-Solar like

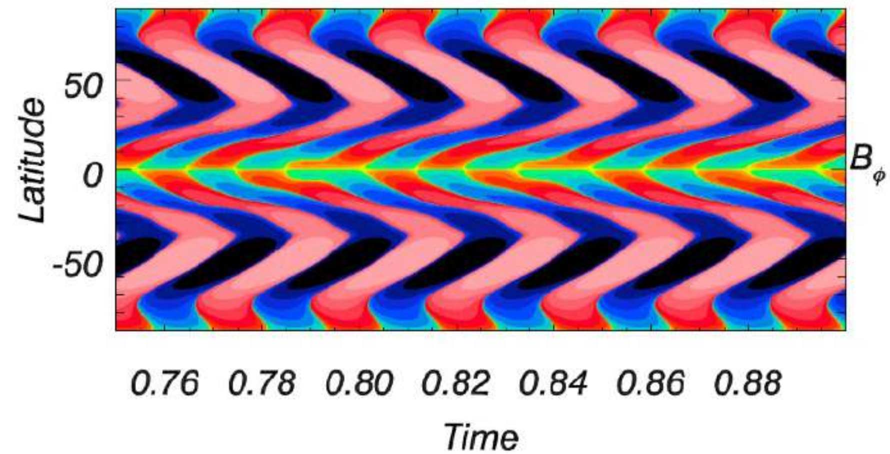


Figure 2.21 – Solar and anti-solar butterfly diagrams (time-latitude contour plot) of an α-ω dynamo models. Shown are color contours of the amplitude of the toroidal field near the base of the convection zone with red tones denoting positive polarities. Left: Solar case model with prograde dynamo waves for the low latitude band. the model was computed with ($C_\alpha = -10$) and $C_\Omega = 2.5e4$)). Right: Same case with a reverse sign for the α-effect ($C_\alpha = +10$)). Note the change of direction of the dynamo wave.

Summary dynamo solutions using mean field approach

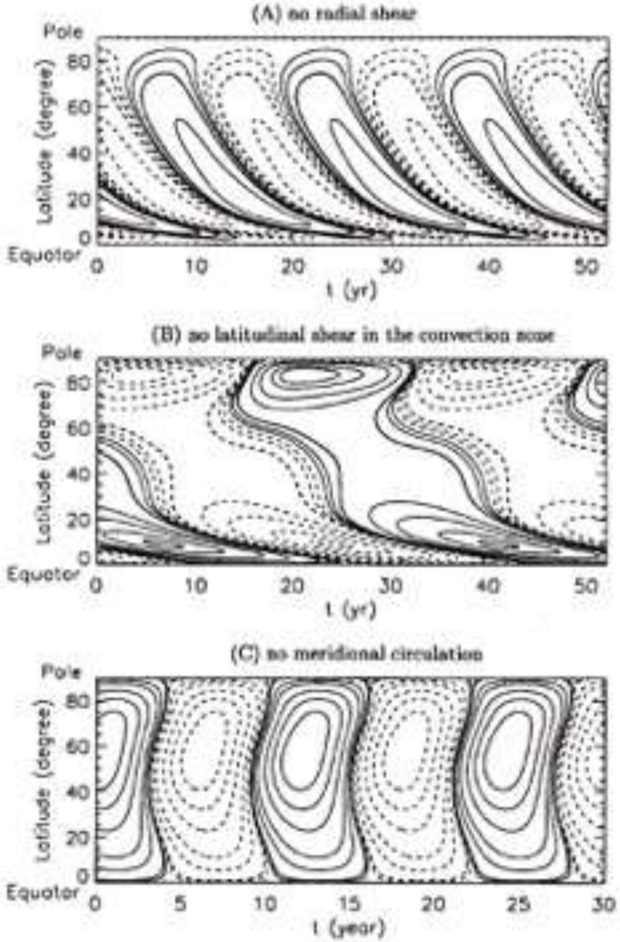


FIG. 4—Three toroidal field butterfly diagrams resulting from various numerical "surgical" experiments. The format is the same as in Fig. 3*a*. (*a*) Solution where the radial shear was artificially shut off, with only the latitudinal shear left to contribute to the generation of toroidal fields. (*b*) Opposite experiment, i.e., the latitudinal shear has been artificially shut off. For these two solutions all parameter values are otherwise identical to the reference solution of Figs. 2 and 3. (*c*) Solution where the meridional circulation has been turned off. The resulting butterfly diagram bears a striking resemblance to that produced by mean field interface dynamos (see text).

Diagramme papillon

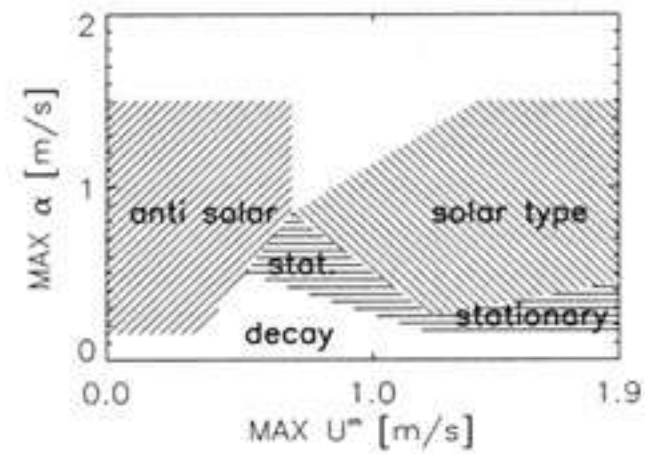


Fig. 11. The different types of solution found for varying strengths of the α-effect and the flow speed. The terms *solar type* and *anti solar* refer to equatorward and poleward drifting field belts, respectively, while *stationary* refers to a stationary field. The magnetic diffusivity always has a value of 10^{11} cm^2/s.

(Charbonneau et Dikpati 2001, Kuker et al. 2002)

Mean field dynamo models (pros & cons)

- Solve the induction equation in axisymmetric form
(thus introducing an extra term to satisfy Cowling's anti-dynamo theorem)

- This is the alpha effect, toroidal -> poloidal
or a surface source term S, toroidal -> poloidal

- Assume a rotation profile => Omega effect (pol->tor)

- Kinematic regime (no feedback on velocity field)

Some models also prescribe meridional circulation

Advantages: Fast (broad study of parameter space)
Alpha parameterization allows fine tuning
Results comparable to observations
Disadvantages: kinematic, prescribed alpha, Omega (possibly deduced from helioseismology), MC (if present) and magnetic diffusivity often in an inconsistent manner

Magneto-Hydro-Dynamic (MHD) equations

Continuity, Navier-Stokes, Energie (+Lorentz force + Ohmic dissipation): :

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) , \quad \boxed{\nabla \cdot \mathbf{B} = 0}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla P + \rho \mathbf{g} - 2\rho \mathbf{\Omega_o} \times \mathbf{v}$$

$$- \nabla \cdot \mathcal{D} + \boxed{\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}},$$

$$\rho T \frac{\partial S}{\partial t} = -\rho T(\mathbf{v} \cdot \nabla)S + \nabla \cdot (\kappa_r \rho c_p \nabla T) + \boxed{\frac{4\pi\eta}{c^2}\mathbf{J}^2}$$

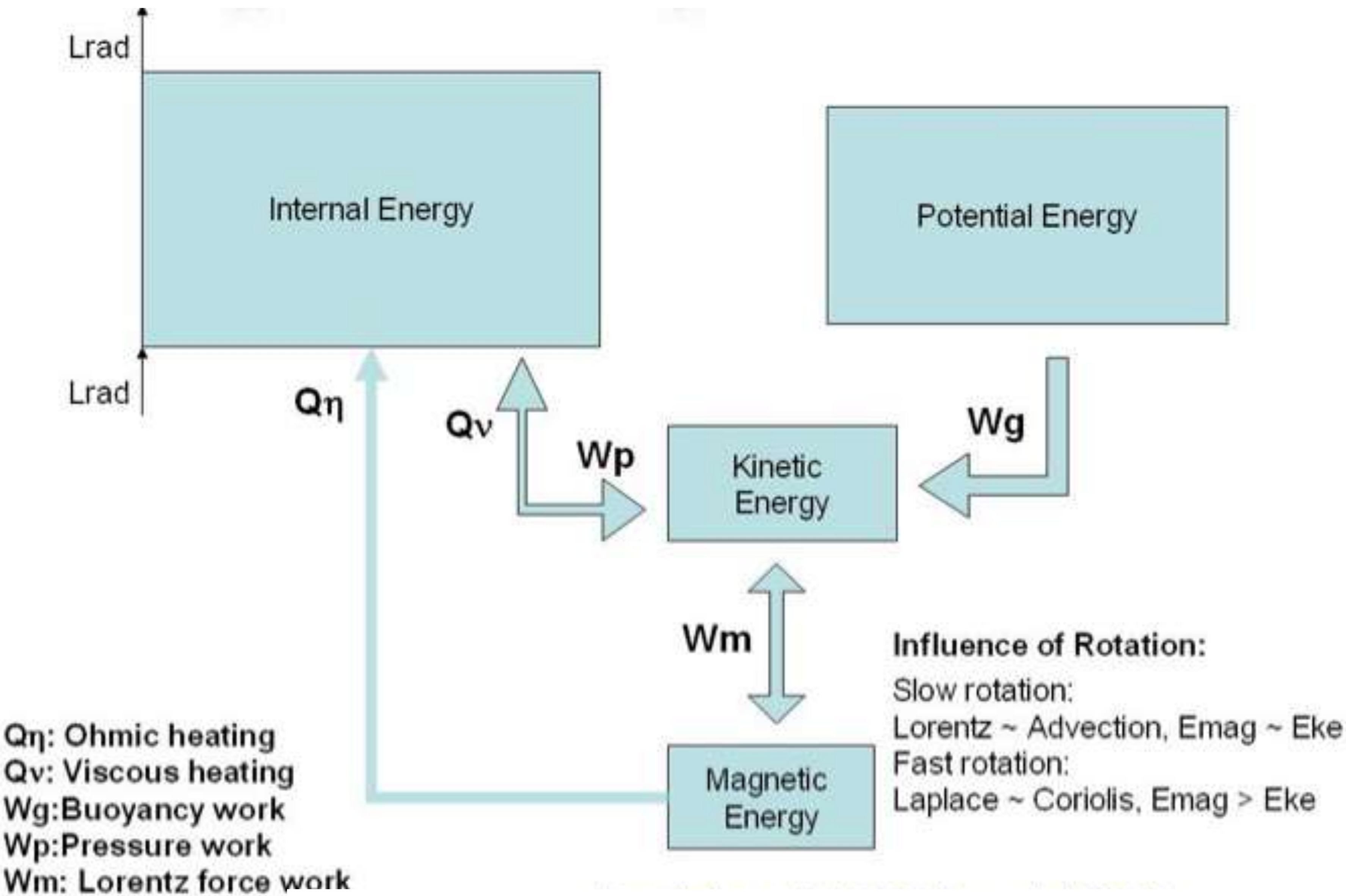
$$+ 2\rho\nu \left[e_{ij}e_{ij} - 1/3(\nabla \cdot \mathbf{v})^2\right] + \boxed{\rho\epsilon},$$

Volumetric heat source (nuclear energy)

plus induction:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) \qquad (8)$$

Dynamo: what energy reservoir?



Saar & Gilman 1966, Brandenburg et al. 1996, Brun et al. 2004, 2022 ApJ

Kinematic vs dynamical (non linear) dynamo

If the Lorentz force can be neglected in the Navier-Stokes equation, then we refer to a kinematic dynamo, and the instability is linear with exponential growth.

Otherwise (which occurs for B fields of finite amplitude), we refer to a dynamic dynamo, there is feedback from the Lorentz force on the plasma motions, the instability saturates, and the magnetic field B reaches a finite amplitude. The magnetic energy ME=$B^2/8\pi$ is close to equipartition with the kinetic energy KE=$0.5\rho v^2$ of fluid motions (depends on parameters regime).

Remark: the Lorentz force can be decomposed in 2 parts:

$$\mathbf{F}_L = \frac{1}{c}\mathbf{J}\times\mathbf{B} = \frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}$$

$$= \underbrace{-\boldsymbol{\nabla}\mathbf{B}^2/8\pi}_{\text{Pression}} + \underbrace{(\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{B}/4\pi}_{\text{Tension}}$$

Dynamo regime and simple scaling laws

Equilibrium field: $B_{eq} \sim (8\pi P_{gaz})^{1/2} \sim \rho^{1/2}$

Laminar scaling (weak), Lorentz ~ diffusion ($Rm \sim 1$, $v = \eta/L$):

$$B^2_{\mathrm{weak}} \sim \rho \nu \eta / L^2$$

Turbulent scaling (equipartition), Lorentz ~ advection:

$$B^2_{\mathrm{turb}} \sim \rho v^2 \sim \rho \eta^2 / L^2 \Leftrightarrow |B_{\mathrm{weak}}| \sim |B_{\mathrm{turb}}| P_m^{1/2}$$

Magnetostrophic scaling (strong), Lorentz ~ Coriolis:

$$B^2_{\mathrm{strong}} \sim \rho \Omega \eta$$

Where ρ is density, ν is kinematic viscosity, η is magnetic diffusivity, Ω is the rate of rotation, v and L are the characteristic velocity and size, and $Pm = \nu/\eta$ is the magnetic Prandtl number.

Christensen 2010, Davidson 2013, Brun et. 2015, Augustson et al. 2019

Nonlinear dynamo models (pros & cons)

- Solve MHD equations (therefore no extra terms)

- Dynamic regime (feedback on the velocity field)

- Models with or without convection

- Certain models can prescribe specific aspects of stars (shear in the tachocline, boundary conditions, etc.)

Advantages: Self-consistent, ab initio physics, flows compatible with each other (Omega, MC, convection), 3D (longitude dependence, for example)

Disadvantages: slow (because it is complicated), so only a small study of parameters for the moment, no complete model to date, comparison with observations less direct, difficult to place within a stellar parameter regime

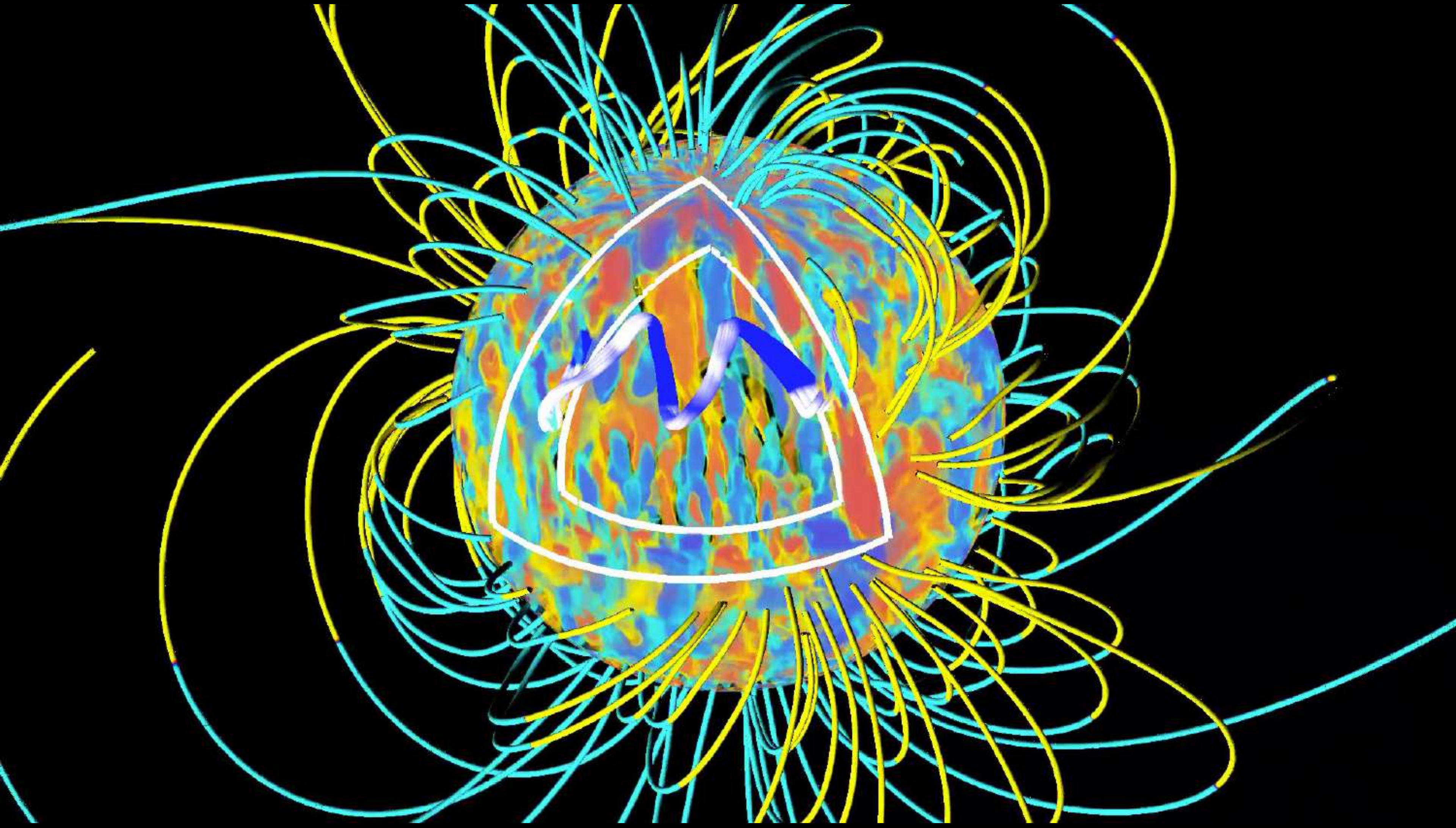
Lecture plan

I. Energy transport in stars

II. Stellar fluid dynamics
- From particle description to fluid description in plasmas
- Turbulence: basic concepts
- Convection simulation: some exemples

III. Dynamo effect in turbulent convective zones of stars
- The Sun's magnetic field
- The dynamo effect: fundamental ingredients
- The dynamo effect: kinematics vs. dynamics
- Stellar magnetism and dynamo

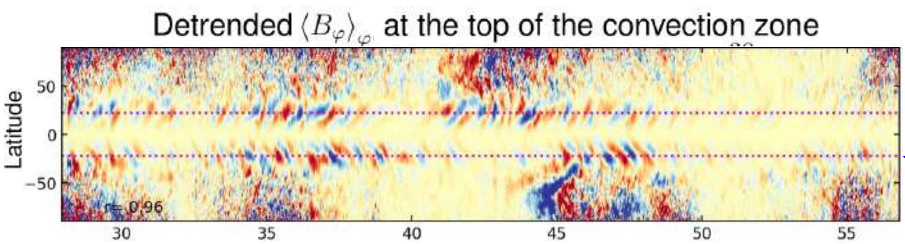
Recent advances on nonlinear cyclic dynamos

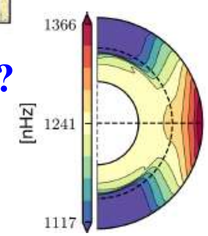


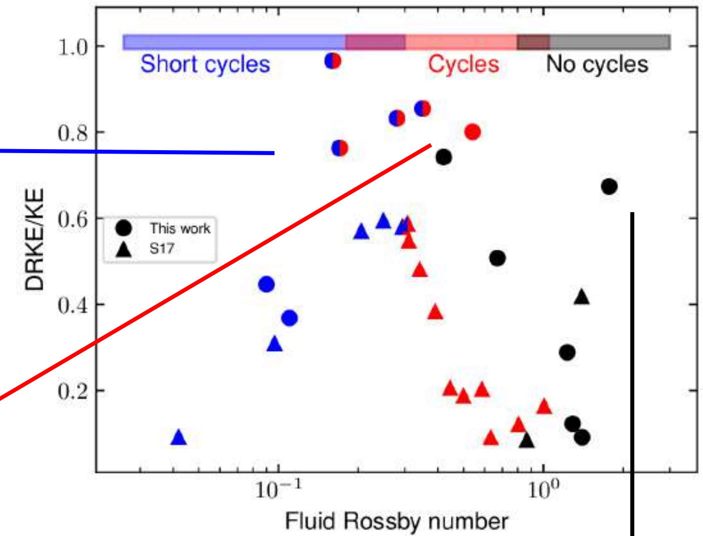
Strugarek et al. 2017 (Science)

Diversity of stellar dynamos vs Rossby number

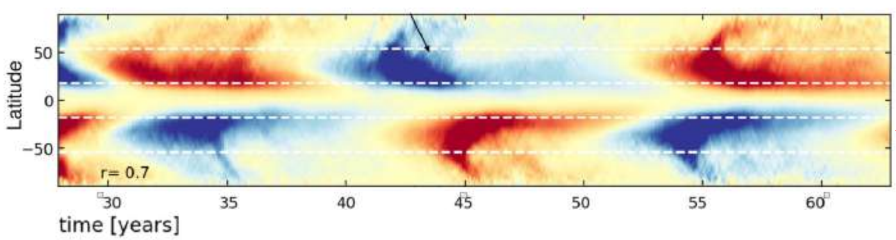
Rotational and magnetic transitions







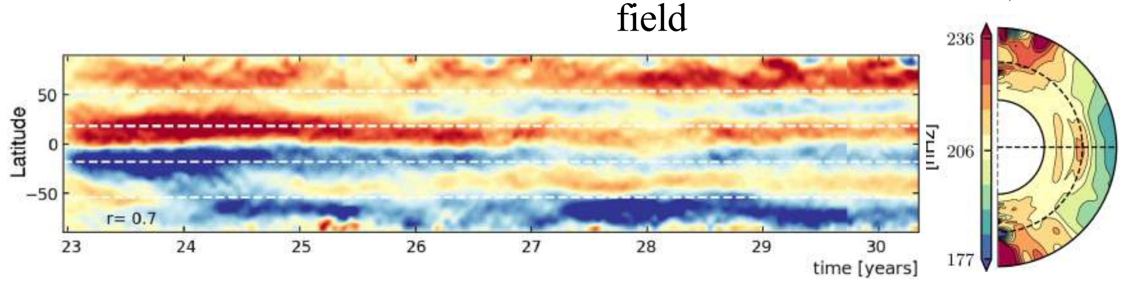
- short cycles (~year) ; **quasi-biennial oscillations ?**
- surface dynamo
- Parker-Yoshimura type (αΩ)



$\langle B_\varphi \rangle_\varphi$ at the base of the convection zone

- long cycles (**decadal solar-like**)
- deeper dynamo
- non-linear retroaction

- **Stationnay** dynamo
- hemispherical toroidal field



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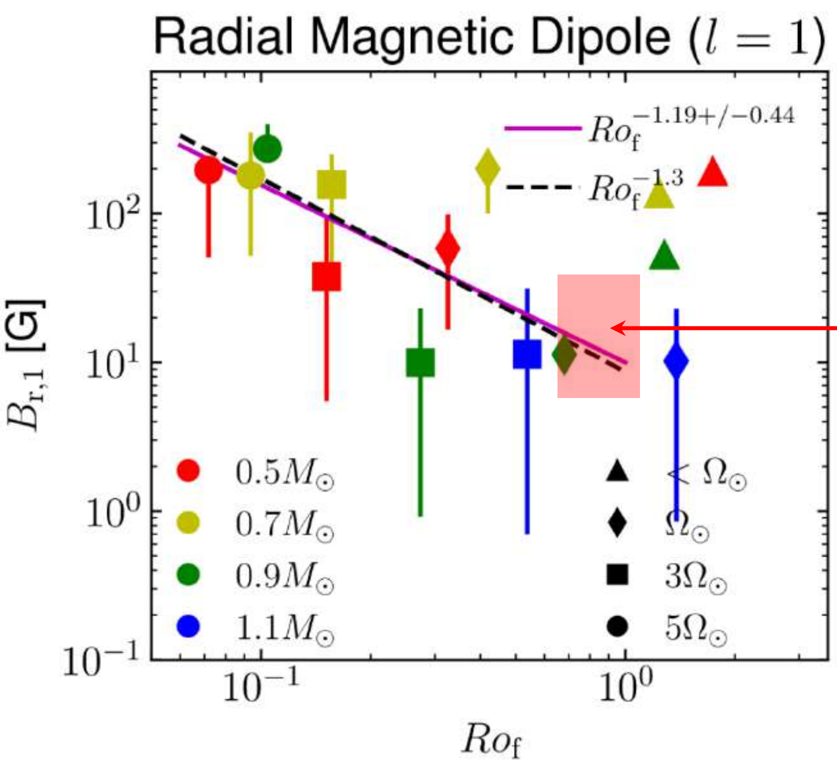
Recovering Observational Trends

Weakened magnetic braking?



Brun, Strugarek, Noraz+ 22

- The dipole decreases but does not disappear

However, there may be a minimum around the solar Rossby value

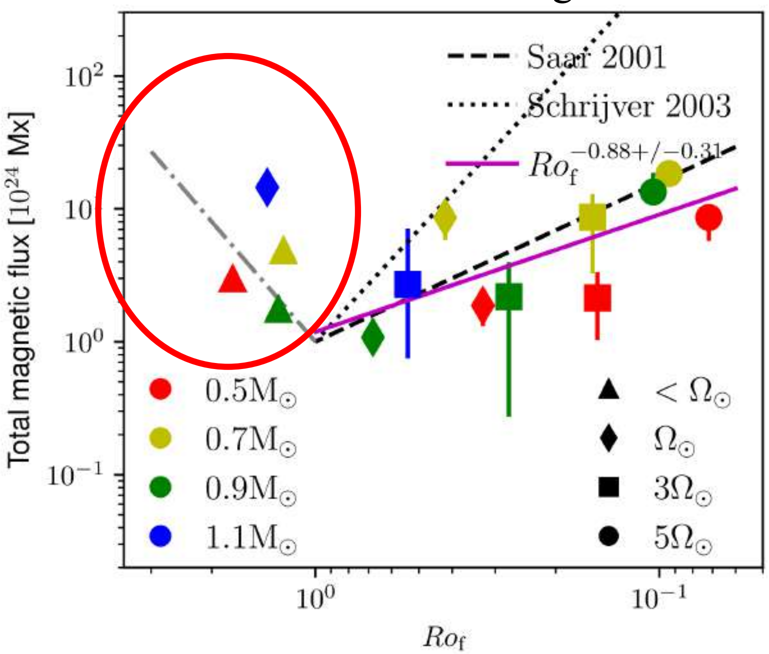
- Can a star be trapped in this regime by a **combination of magnetism and mass loss rate?**

$$\dot{J} \propto \dot{M}\Omega_* \langle r_A \rangle^2$$

Recovering Observational Trends

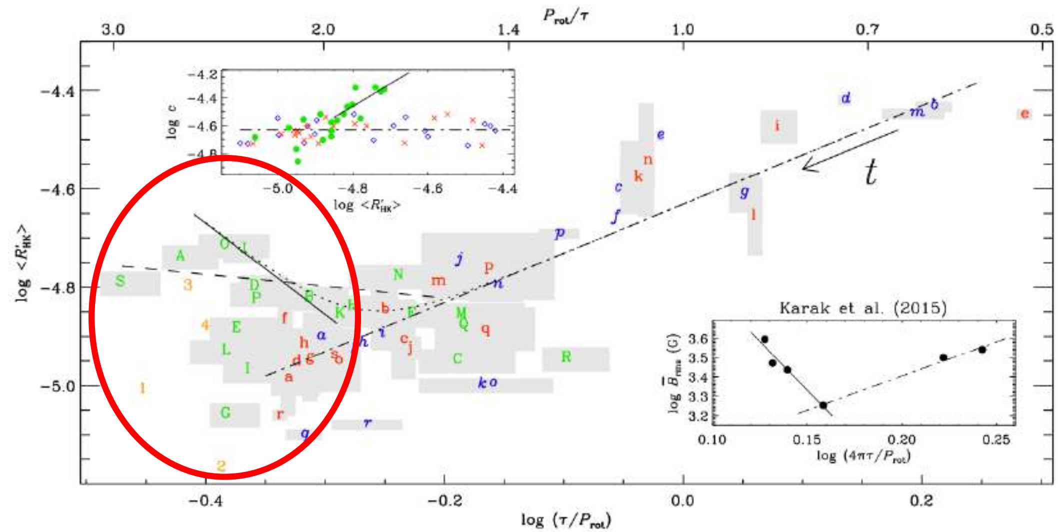
Magnetic behavior in the high Rossby regime

Possible increase of the magnetic flux

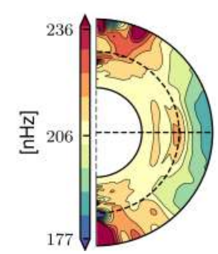


Brandenburg & Giampapa 18

Possible enhancement of the activity



- Need **theoretical** and **observational constraints** to enlighten what occurs in the anti-solar regime



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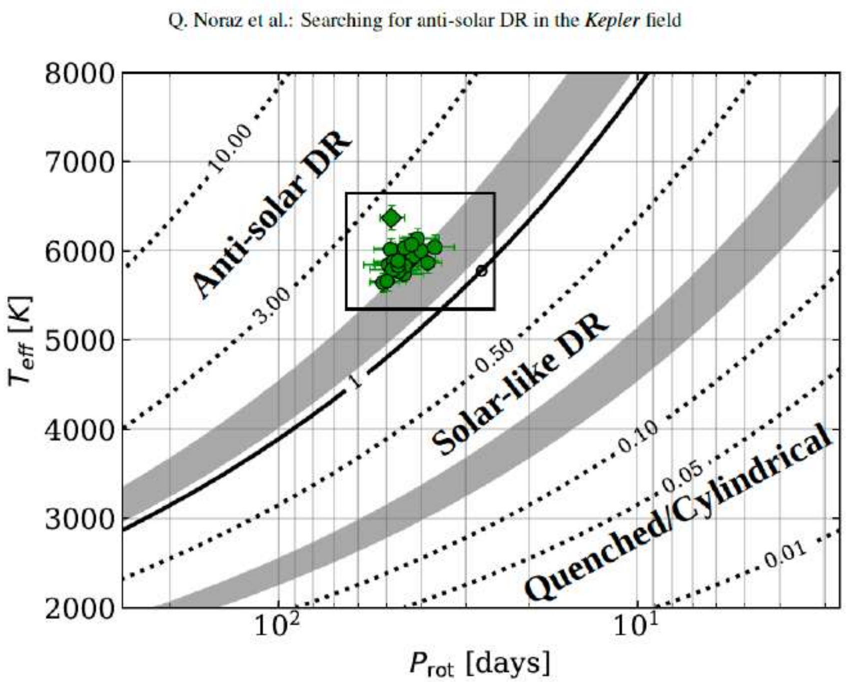
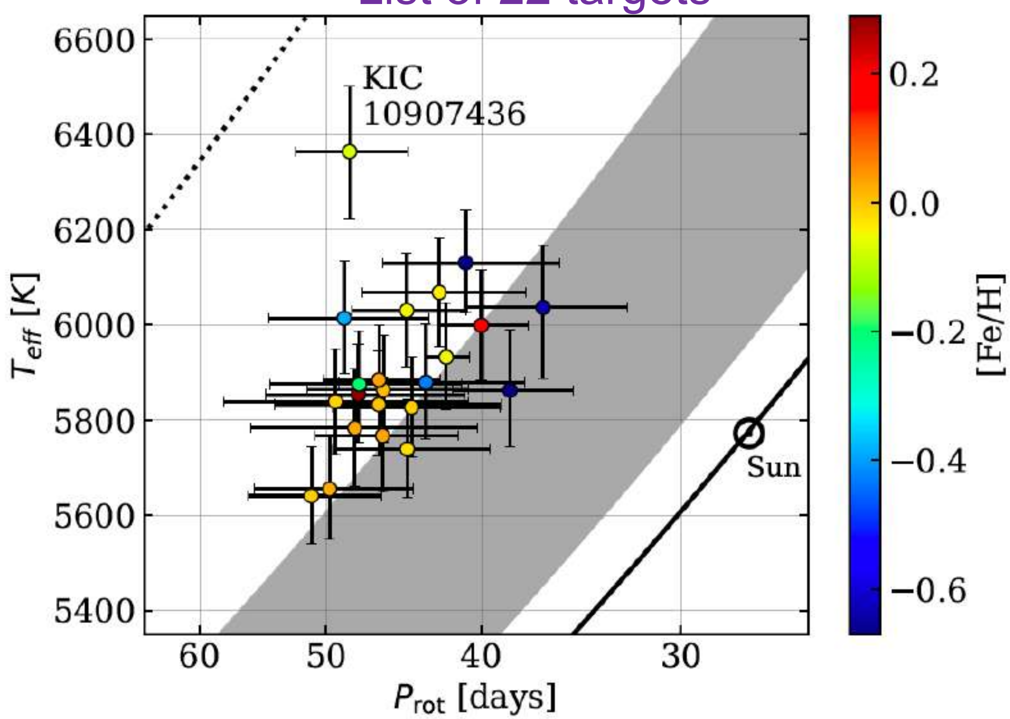
Searching for anti-solar stars in the Kepler field

Adapting theoretical Rossby to Observational Rossby



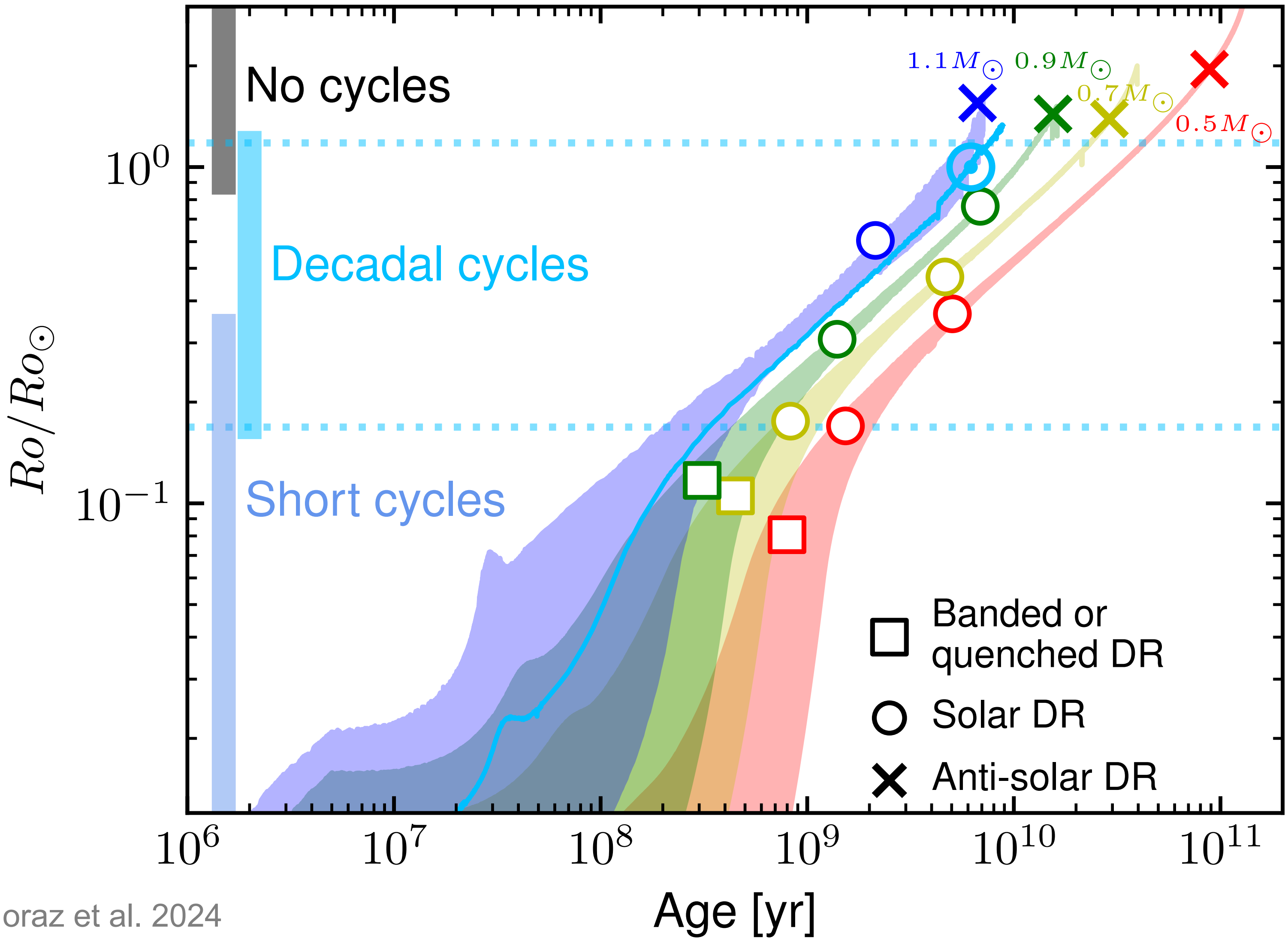
List of 22 targets



$$\frac{Ro_f}{Ro_{f,\odot}} = \left(\frac{P_{rot,*}}{P_{rot,\odot}}\right) \times \left(\frac{T_{eff}}{T_{eff,\odot}}\right)^{3.29} \times \left(\frac{[Fe/H]+2}{2}\right)^{-0.31}$$

Noraz et al. 2022b

A plausible « Sun in time » story



Noraz et al. 2024

Next Lecture:

Helio and Asteroseismology