Fluid dynamics, vortex dynamics, and model building

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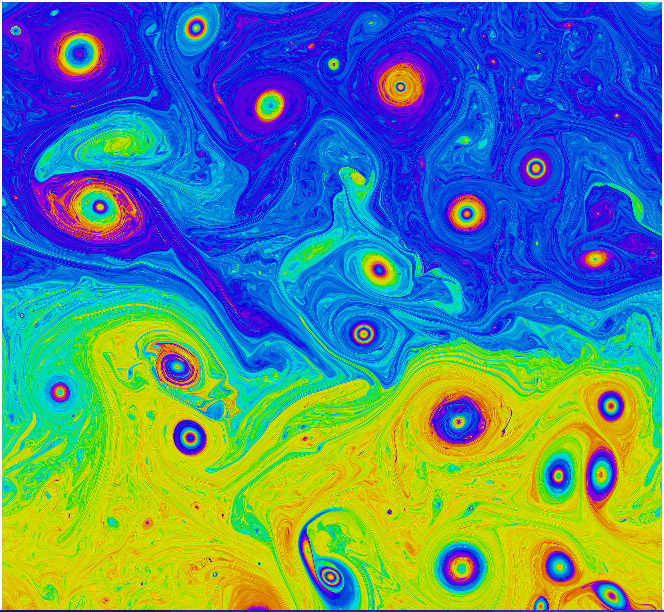
Outline

The Great Red Spot

Synthetic (simulated) spots

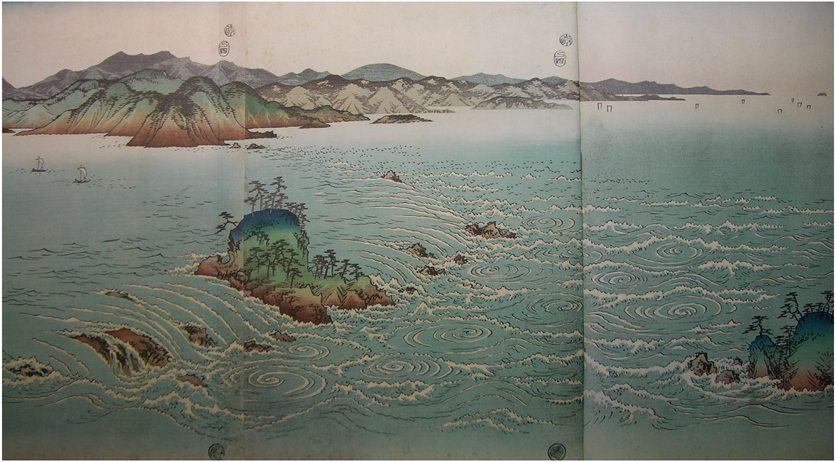


A cloud

A synthetic (simulated) cloud

Rendered image by Domantas Dilys, University of Leeds

In art

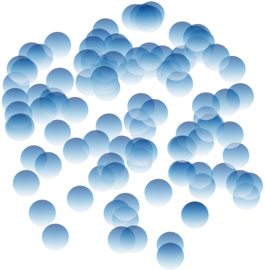


Seascape at Naruto, Awa ('Awa Naruto no Fuukei' in Japanese),
by Andô Hiroshige (1857).
Image courtesy of Professor Mitsu Funakoshi, Kyoto University.

Awa Province: Wind and Waves at the Whirlpool of Naruto
by Andô Hiroshige (1855).

What is a fluid?

Air and water are common examples of a fluid
Fundamentally, both are made up of a collection of rapidly-moving molecules at very small scales.



In fact temperature is defined to be proportional to the mean kinetic energy of the molecules.

The key point is that, while typically molecules are separated by large empty spaces, they are *so numerous* ($\sim 10^{20}$ per cm^3) that it is practically impossible to follow each molecule to deduce their collective motion.

... The fastest computer in the world could not simulate the motion of this many molecules!

Fluid "motion"

Fluid dynamics is concerned with the *mean* motion of these molecules over tiny volumes, indeed as we take the volumes to zero (here we become mathematicians!).

Of course, meaningful averages can be taken only if the tiny volumes still contain a large number of molecules. A cubic micrometer μm^3 of air still contains 2.7×10^{10} molecules....

Molecules move at high speeds in different, almost random directions. When you average these speeds, there is a **huge** amount of cancellation. That is, in the x direction, some molecules will have a positive speed while others will have a negative one, and averaging over all the speeds leads to a mean speed which is *much* less than the typical speed of each molecule.

A (very) brief history

The *mathematical* basis of fluid dynamics was not formulated until the mid 1700's, in the pioneering work of Euler, Bernoulli, Navier, and Stokes.

The first complete set of equations was derived by Leonhard Euler in 1757, in "Principes généraux du mouvement des fluides". They take the form

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \left(= \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right) = -\frac{\boldsymbol{\nabla} p}{\rho} + \boldsymbol{F}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

— for *uniform* density ρ. Here $\boldsymbol{F}$ stands for external forces (e.g. gravity).

The first equation above is an expression of Newton's famous law $F = ma$, but for a continuum. What was truly radical was the inclusion of pressure p, the force per unit area exerted by molecular collisions.

Adding complexity ...

Real fluids have *variable* density ρ, though in some cases (like the oceans) it varies only slightly (albeit with important consequences). Conservation of mass generally requires

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0\,.$$

Moreover, real fluids are *compressible* (permit sound waves) yet can often be usefully approximated as incompressible:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \Rightarrow \quad \frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$$

— then density is *materially conserved*.

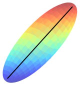
The governing equations for a compressible fluid are much more complicated: they require an equation of state relating density, pressure and temperature, and further equations for energy and entropy.

Vorticity — the "spin" in a fluid

A century after Euler published his equations, Hermann von Helmholtz derived his famous vortex theorems in 1858. These refer to the properties of vorticity

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$$

— a vector quantity that gives twice the local spin of an infinitesimally small fluid volume.



This changes by s t r e t c h i n g and buoyancy effects.

The importance of vorticity is that it tends to be highly localised in fluid flows. One can often readily identify *"vortices"* in naturally-occurring flows.

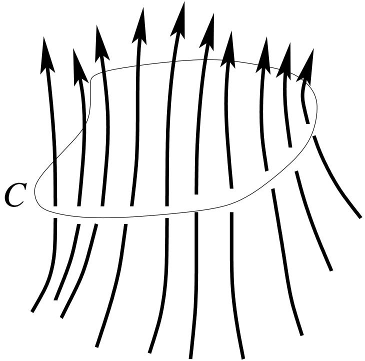
They are, indeed, ubiquitous throughout the universe.

Circulation and Potential Vorticity

A decade *after* Helmholtz published his vortex theorems, Lord Kelvin derived 'his' circulation theorem. This states that the flux of vorticity through any *material* surface,

$$\iint_S \boldsymbol{\omega} \cdot \hat{\boldsymbol{n}}\, dS = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}\,,$$

is constant in time for Euler's equations.



Beltrami and Material Vorticity

Now suppose that the material surface S is described by $\theta(\boldsymbol{x}, t) = \theta_0$, a constant, i.e.

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0$$

(material points generally move on this surface, but retain $\theta(\boldsymbol{x}, t) = \theta_0$).

Shrinking C to a point and using $\hat{\boldsymbol{n}} = \boldsymbol{\nabla}\theta / |\boldsymbol{\nabla}\theta|$, Kelvin's theorem implies

$$\frac{\mathrm{D}(\boldsymbol{\omega} \cdot \boldsymbol{\nabla}\theta)}{\mathrm{D}t} = 0$$

pointwise. Alternatively, this may be derived directly from the vorticity equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\boldsymbol{u}\,.$$

Beltrami and Material Vorticity

Euler's equations have the property that the curl of the fluid acceleration $\boldsymbol{a} = \mathrm{D}\boldsymbol{u}/\mathrm{D}t$ is identically zero, i.e.

$$\boldsymbol{\nabla} \times \boldsymbol{a} = \mathbf{0}\,.$$

Beltrami (1871) proved that, under this condition, the quantity

$$\varpi = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\theta$$

is materially conserved on a θ surface (where $\mathrm{D}\theta/\mathrm{D}t = 0$). This quantity, ϖ, is Beltrami's material vorticity — see Á. Viúdez, "The Relation between Beltrami's Material Vorticity and Rossby-Ertel's Potential Vorticity", J. Atmos. Sci. **58** (2001).

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However, Euler's equations don't apply to atmospheric and oceanic fluid dynamics, where density ρ varies and where the fluid in not strictly incompressible.

Nonetheless, Beltrami's theorem does apply whenever

$$\nabla\theta \cdot (\nabla \times \boldsymbol{a}) = 0\,,$$

which does commonly occur in atmospheric and oceanic fluid dynamics (ignorning diabatic and frictional processes).

Consider incompressible flow (for which $\nabla \cdot \boldsymbol{u} = 0$ and $\mathrm{D}\rho/\mathrm{D}t = 0$) having variable density ρ. In a rotating frame of reference, with vector rotation $\boldsymbol{\Omega}$, the velocity field satisfies

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{x} = -\frac{\nabla p}{\rho}\,. \tag{1}$$

Potential Vorticity

The left-hand side of this equation is the acceleration $\boldsymbol{a}$ in the absolute (inertial) reference frame. Hence,

$$\nabla \times \boldsymbol{a} = \frac{\nabla\rho \times \nabla p}{\rho^2}\,. \tag{2}$$

It therefore follows that $\nabla\rho \cdot (\nabla \times \boldsymbol{a}) = 0$. Moreover, $\rho =$ constant is a material surface, on account of $\mathrm{D}\rho/\mathrm{D}t = 0$.

Hence, by Beltrami's theorem, the quantity

$$q = \boldsymbol{\omega}_a \cdot \nabla\rho$$

is materially conserved: $\mathrm{D}q/\mathrm{D}t = 0$. Here, $\boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega}$ is the absolute vorticity. More generally, $q = \boldsymbol{\omega}_a \cdot \nabla F(\rho)$ is materially conserved for any *functional* F.

Potential Vorticity

The quantity q is known as *potential vorticity*, later (re)derived by Ertel (1942), generalising a result of Rossby (1940).

In a compressible atmosphere, ρ is no longer conserved, however the entropy, or (equivalently) potential temperature $\theta = T(p_{\mathrm{ref}}/p)^{2/5}$ is. In this case,

$$q = \frac{\boldsymbol{\omega}_a \cdot \boldsymbol{\nabla}\theta}{\rho}$$

is materially conserved — again by Beltrami's theorem.

The concept of potential vorticity (PV) is **hugely** important in atmospheric and ocean dynamics, see Hoskins, McIntyre & Robertson, Quart. J. Roy. Meteorol. Soc. (1985).

Quart. J. R. Met. Soc. (1985), **111**, pp. 877–946 551.509.3:551.511.2:551.511.32

On the use and significance of isentropic potential vorticity maps

By B. J. HOSKINS[1], M. E. McINTYRE[2] and A. W. ROBERTSON[3]

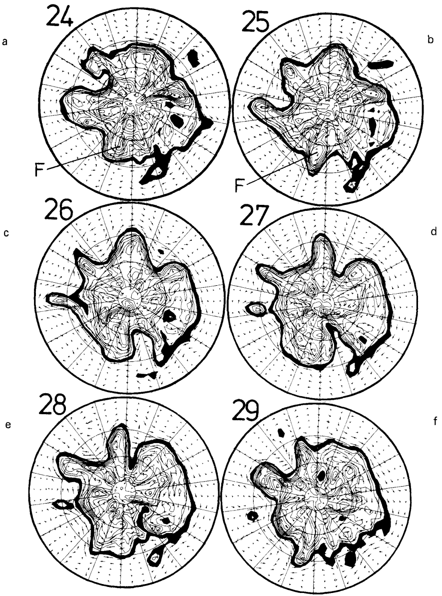


Figure 3. 330 K IPV maps for the region north of 20°N for the period 24–29 September 1982. The contour interval is 1·0 PV units and the regions with values 1–2 units are blacked in. Also shown are the horizontal velocity vectors on this surface, scaled as in Fig. 2.

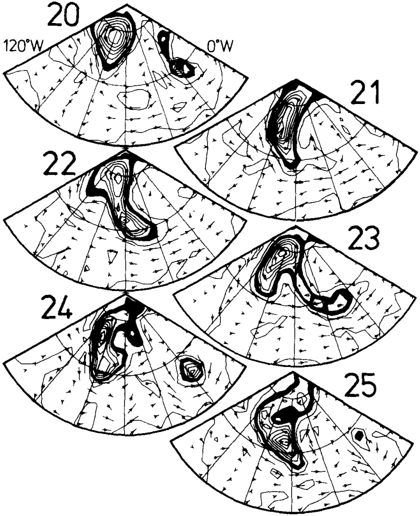
The stratospheric polar vortex.

Here the analysed (isentropic) PV is shown at successive days in September 1982.

The edge of the polar vortex is what we identify as the "jet stream".

In the Earth's stratosphere, contours of PV are virtually *material contours*, meaning they carry the same fluid particles at all times (if q is conserved).

PV is not passive, but directly feeds back into the flow and **largely** controls the flow's dynamical and thermodynamical structure through *"PV inversion"*.



's of the 300 K IPV maps for the period 20–25 September 1982. The region (

Quart. J. R. Met. Soc. (1985), **111**, pp. 877–946

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On the use and significance of isentropic potential vorticity maps

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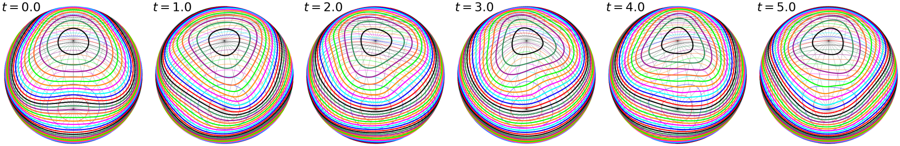
Rossby and inertia–gravity waves

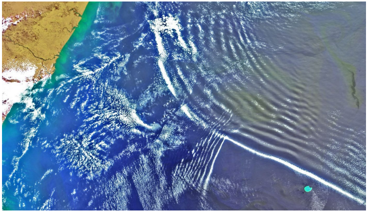
In general, waves rely an *underlying restoring mechanism* (a medium through which they can propagate) for their existence.

The most important waves (i.e. most energetic) in geophysical flows are *Rossby waves* and *inertia–gravity waves*.



Rossby waves propagate on the *variable planetary vorticity* associated with planetary rotation.

Inertia–gravity waves



Inertia–gravity waves propagate on both vortex lines (associated with background rotation) and variations in mean density.

- Rotational waves propagating on vortex lines alone are called *inertial waves*.

- Density waves propagating on variations in the mean density alone are called *internal waves*.

Balance and PV inversion

How can we exploit PV conservation? We shall see that PV is the key field associated with the underlying, slowly evolving, balanced flow.

PV is not enough to determine all the dynamical and thermodynamical fields. E.g. in the incompressible equations, there are four "prognostic" equations (with time derivatives): those for $\boldsymbol{u}$ and ρ.

The condition $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ eliminates one, but the three remaining prognostic equations cannot be replaced simply by $\mathrm{D}q/\mathrm{D}t = 0$.

What pair of fields should one evolve alongside PV?

Can we exploit any approximate relationships between fields to reduce the equations to PV conservation plus diagnostic relations? (Yes!)

The barotropic model: two-dimensional (2D) flow

To illustrate, we start with the simplest flow relevant to large-scale atmospheric and oceanic dynamics: 2D flow on the plane governed by Euler's equations.

Since $w = 0$ and $\partial F/\partial t = 0$ for any field $F(x, y, t)$, the vorticity equation reduces to scalar conservation of vertical vorticity ζ:

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0$$

(This also follows directly from Kelvin's circulation theorem).

The definition $\boldsymbol{\omega} = (\xi, \eta, \zeta) = \boldsymbol{\nabla} \times \boldsymbol{u}$, and $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, together imply

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \zeta \qquad \text{and} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Two-dimensional (barotropic) flow

One may thus "invert" ζ by introducing a streamfunction ψ in terms of which

$$u = -\frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = \frac{\partial \psi}{\partial x} \,.$$

It follows that

$$\nabla^2 \psi = \zeta \,.$$

Hence, given $\zeta(x, y, t)$ (our materially-conserved PV), the flow field $\boldsymbol{u}(x, y, t) = (u, v)$ is found by inverting the Laplacian ∇^2 above and differentiation.

With $\boldsymbol{u}$ thus found, ζ can be propagated to the next instant of time, and so on. This is a fully self-contained system — already a balanced model.

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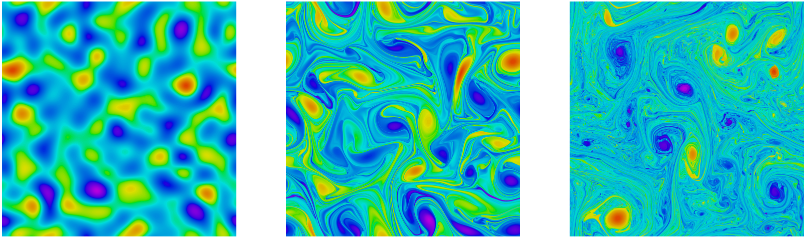
We'll see that this is closely related to the important Quasi-Geostrophic balance model in Geophysical Fluid Dynamics (GFD).

Two-dimensional (barotropic) flow

Appropriate efficient numerical methods will be discussed later.

Suffice it to say, 2D flow is capable of exhibiting extraordinary complexity:



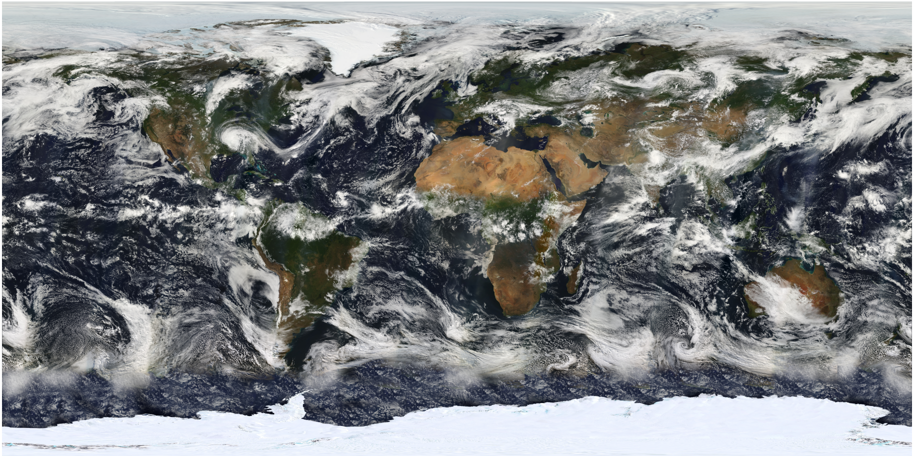
Vortices spontaneously arise and grow through merger (on average).

The energy distribution across scales — the spectrum — cascades to large scales, exactly opposite to what is found in 3D!

See Dritschel *et al* in *J. Fluid Mech.* **640**, 215–233 (2009).

Turbulence and self-organisation

Collectively, geophysical fluids display a remarkable degree of organisation.



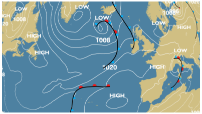
Turbulence is *not* simply chaotic motion; it is *not* without structure.

Order out of chaos … or *no* chaos?

How do we start modelling something this complex?

Modelling is much more than (approximately) solving the governing mathematical equations on large computers.

This is necessary and important for some applications (e.g. weather forecasting), but it is only part of the story.



For a mathematician, a **model** represents a simplification of a complex process, a simplification that leads to a deeper and clearer understanding.

The idea is to retain the key ingredients involved and remove those which make analysis very difficult or impossible.

Modelling in action

In fact, we may be unaware that we are modelling all the time! It is a deeply psychological concept.

We naturally gloss over detail and pick out key features.

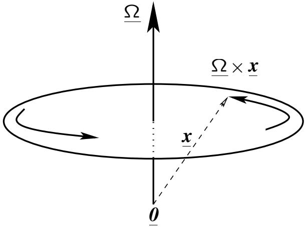
◇ ◇ ◇ *We are programmed to do this.* ◇ ◇ ◇

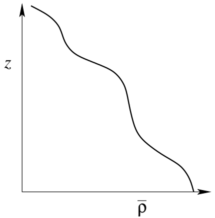
Mathematicians and scientists generally formalise this in their "research". We use approximate equations to describe weather, ignoring effects (e.g. quantum) which are either negligible or obstructive.

Simpler models enable us to identify what essentially controls a given process — like the growth of a cloud.

Recipé for geophysical fluid dynamics

The extra and *essential* ingredients needed to create observed patterns in planetary atmospheres and oceans are rotation, stratification and forced thermal variations, e.g. from the sun.

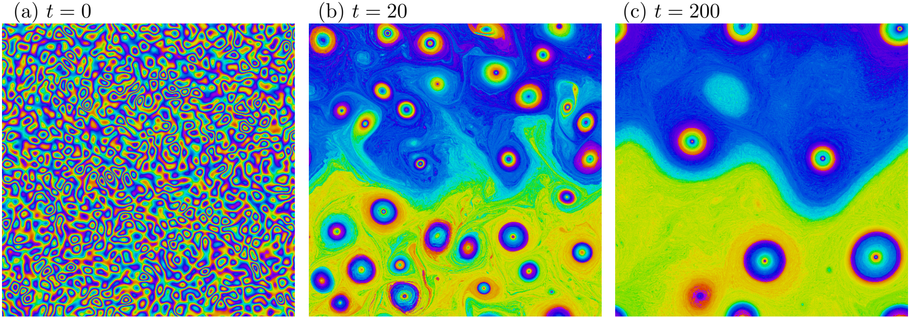




- Stratification leads to *layering* — predominantly horizontal motions.
- Rotation leads to *vertical coherence* — causing different vertical layers to move together.
- Thermal variations cause *baroclinic instability* and *convection*, leading to *coherent patterns* (jets, vortices, fronts) in a *turbulent 'sea'*.

The process of order formation is *generic* in geophysical flows.

It arises from the *inhomogeneous mixing* of an important dynamical tracer, potential vorticity (PV), closely related to Kelvin's circulation.



Here, jets and vortices form in a single-layer quasi-geostrophic flow. Shown is the evolution of the PV field (from Scott & Dritschel, in *Zonal Jets*, Cambridge University Press, 2019).

How do we start modelling?

We *reduce* more complicated models (there is no complete model!) to simpler, more manageable forms by neglecting terms (or whole equations). This is often guided by *scale analysis*.

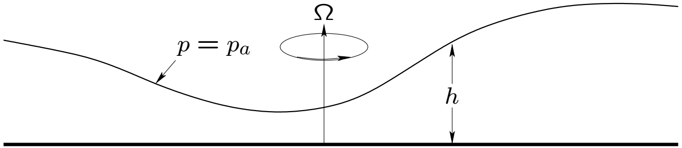
In the scaled, dimension-*less* equations, one or more small parameters appear. One may then derive a reduced model through an asymptotic expansion.

Alternatively, one may make an *ansatz* that *a priori* some condition holds, e.g. there is a symmetry like the flow is independent of one spatial coordinate.

Both techniques are illustrated next.

From shallow water to quasi-geostrophy

A very common model used to study geophysical flows is the shallow-water model. In its simplest form, it describes the motion of a *homogeneous* (uniform density) fluid with a free surface, held down by gravity:



Here, we include the background (planetary) rotation $\Omega = f/2$.

The shallow-water (SW) model is predicated upon the *hydrostatic approximation*, which states that the vertical acceleration $\mathrm{D}w/\mathrm{D}t$ is *negligible* compared with gravity g. This leads to a balance between the vertical pressure gradient and gravity:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = g$$

— a result going back to Archimedes (287–212 BC)!

From shallow water to quasi-geostrophy

Consistently, the SW model assumes that the horizontal flow (u, v) in (x, y) is independent of z. This leads directly to

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -g\frac{\partial h}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -g\frac{\partial h}{\partial y}, \qquad \frac{\partial h}{\partial t} + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0$$

from the three-dimensional (3D) parent equations. Here, $f = 2\Omega$ is the Coriolis frequency.

Notably, this is a set of coupled nonlinear PDEs, *but only in 3 variables* (u, v, h) depending on (x, y, t). Above $\boldsymbol{u} = (u, v)$ is the horizontal (2D) velocity field and $\boldsymbol{\nabla}\cdot$ is the 2D divergence operator.

The vertical velocity w is a linear function of z but is *purely diagnostic*:

$$w = -z\,\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

From shallow water to quasi-geostrophy

To derive the quasi-geostrophic (QG) model, we perform a scale analysis to reveal small parameters.

First of all, mass conservation means that the mean height $H = \langle h \rangle$ is conserved. We consider therefore the *displacement* $\eta = h - H$.

Let L, D, U and T be characteristic length, displacement, velocity and time scales. Anticipate/assume that the time scale for the flow evolution is the *advective* one:

$$T = \frac{L}{U} .$$

This means that $\partial/\partial t \sim \boldsymbol{u} \cdot \boldsymbol{\nabla}$.

Begin by rewriting the equations using dimensionless variables:

$$\tilde{t} = t/T , \quad \tilde{\boldsymbol{x}} = \boldsymbol{x}/L , \quad \tilde{\boldsymbol{u}} = \boldsymbol{u}/U \quad \text{and} \quad \tilde{\eta} = \eta/D$$

From shallow water to quasi-geostrophy

$$\frac{U}{fL}\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\tilde{u}\right)-\tilde{v}=-\frac{gD}{fUL}\frac{\partial\tilde{\eta}}{\partial\tilde{x}},$$

$$\frac{U}{fL}\left(\frac{\partial\tilde{v}}{\partial\tilde{t}}+\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\tilde{v}\right)+\tilde{u}=-\frac{gD}{fUL}\frac{\partial\tilde{\eta}}{\partial\tilde{y}},$$

$$\frac{D}{H}\left(\frac{\partial\tilde{\eta}}{\partial\tilde{t}}+\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nabla}}\tilde{\eta}\right)+\tilde{\boldsymbol{\nabla}}\cdot\tilde{\boldsymbol{u}}=0.$$

Here the first two equations have been scaled by fU, and the third by $H/T = HU/L$.

All quantities with tildes are assumed to be $\mathcal{O}(1)$.

Three dimensionless parameter groups appear:

$$\frac{U}{fL}\,,\qquad \frac{gD}{fUL}\qquad\text{and}\qquad \frac{D}{H}\,.$$

From shallow water to quasi-geostrophy

The QG model assumes that the first and the third are small:

$$\text{Ro} \equiv \frac{U}{fL} \ll 1 \quad \text{and} \quad \alpha \equiv \frac{D}{H} \ll 1\,.$$

The first is called the "Rossby number", Ro. When Ro $\ll 1$, the dominant horizontal balance is between the Coriolis force and the pressure gradient, known as "geostrophic balance".

When $\alpha \ll 1$, free surface displacements η are small compared to H.

Together, both consistently show that — to leading order — the flow is *non-divergent*:

$$\tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{u}} = 0.$$

The horizontal equations then reduce to (simply)

$$\tilde{u} = -\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \qquad \text{and} \qquad \tilde{v} = \frac{\partial \tilde{\psi}}{\partial \tilde{x}}$$

where

$$\tilde{\psi} \equiv \frac{gD}{fUL}\tilde{\eta}$$

is a dimensionless streamfunction. Hence, the final parameter group

$$\frac{gD}{fUL}$$

must be $\mathcal{O}(1)$ for consistency. Without loss of generality, we can take this to be 1, and use this to *define* $D = fUL/g$. Then $\alpha \ll 1$ means

$$\alpha = \frac{D}{H} = \frac{fUL}{gH} = \frac{U^2}{gH}\frac{fL}{U} = \frac{\mathrm{Fr}^2}{\mathrm{Ro}} \ll 1 \quad \Rightarrow \quad \mathrm{Fr}^2 \ll \mathrm{Ro} \ll 1$$

where

$$\mathrm{Fr} \equiv \frac{U}{\sqrt{gH}}$$

is called the "Froude number".

From shallow water to quasi-geostrophy

When $\mathrm{Fr} \ll 1$, flow speeds U are small compared to surface gravity wave speeds $c = \sqrt{gH}$.

The final step in deriving the QG model is to include the *next order corrections* to $\tilde{\boldsymbol{u}}$ — the ageostrophic velocity field in the equation for $\tilde{\eta}$.

We proceed by a *Rossby number expansion* (assuming $\mathrm{Fr} \sim \mathrm{Ro}$),

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_0 + \mathrm{Ro}\,\tilde{\boldsymbol{u}}_1 + \ldots, \qquad \tilde{\eta} = \tilde{\eta}_0 + \mathrm{Ro}\,\tilde{\eta}_1 + \ldots$$

At leading order, $\mathcal{O}(\mathrm{Ro}^0)$,

$$\tilde{u}_0 = -\frac{\partial \tilde{\eta}_0}{\partial \tilde{y}} \qquad \text{and} \qquad \tilde{v}_0 = \frac{\partial \tilde{\eta}_0}{\partial \tilde{x}}$$

but η_0 is as yet undetermined.

From shallow water to quasi-geostrophy

At first order, $\mathcal{O}(\mathrm{Ro}^1)$, the $\tilde{u}$ & $\tilde{v}$ equations give

$$\tilde{v}_1 = \frac{\partial \tilde{\eta}_1}{\partial \tilde{x}} + \frac{\partial \tilde{u}_0}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}_0 \cdot \tilde{\boldsymbol{\nabla}} \tilde{u}_0$$

and

$$\tilde{u}_1 = -\frac{\partial \tilde{\eta}_1}{\partial \tilde{y}} - \frac{\partial \tilde{v}_0}{\partial \tilde{t}} - \tilde{\boldsymbol{u}}_0 \cdot \tilde{\boldsymbol{\nabla}} \tilde{v}_0 .$$

while the $\tilde{\eta}$ equation gives

$$\frac{\mathrm{Fr}^2}{\mathrm{Ro}^2} \left(\frac{\partial \tilde{\eta}_0}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}_0 \cdot \tilde{\boldsymbol{\nabla}} \tilde{\eta}_0 \right) + \tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{u}}_1 = 0 .$$

For consistency, therefore, the ratio Fr/Ro must be $\mathcal{O}(1)$ or smaller.

Note that the first term involving $\tilde{\eta}_1$ in both $\tilde{u}_1$ and $\tilde{v}_1$ is *non-divergent*. Hence $\tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{u}}_1$ *only* involves $\tilde{\boldsymbol{u}}_0$, *not* $\tilde{\eta}_1$! ... A miracle occurs!

From shallow water to quasi-geostrophy

Using a few vector calculus identities, together with $\tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{u}}_0 = 0$, we find

$$\tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{u}}_1 = \frac{\partial \tilde{\zeta}_0}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}_0 \cdot \tilde{\boldsymbol{\nabla}} \tilde{\zeta}_0$$

where

$$\tilde{\zeta}_0 = \frac{\partial \tilde{v}_0}{\partial \tilde{x}} - \frac{\partial \tilde{u}_0}{\partial \tilde{y}}$$

is the leading order vertical vorticity component. Defining

$$\tilde{q}_0 \equiv \tilde{\zeta}_0 - \frac{\mathrm{Fr}^2}{\mathrm{Ro}^2} \tilde{\eta}_0$$

then the $\tilde{\eta}$ equation reduces to

$$\frac{\partial \tilde{q}_0}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}_0 \cdot \tilde{\boldsymbol{\nabla}} \tilde{q}_0 = 0$$

— i.e. material conservation of the scalar $\tilde{q}_0$, the QG potential vorticity.

Recall that the leading-order displacement $\tilde{\eta}_0$ is equal to the streamfunction $\tilde{\psi}$ introduced before:

$$\tilde{\eta}_0 = \tilde{\psi}$$

(this comes from the leading-order geostrophic balance).

Hence, we may write the QG potential vorticity (PV for short) as

$$\tilde{q}_0 = \tilde{\nabla}^2\tilde{\psi} - \frac{\mathrm{Fr}^2}{\mathrm{Ro}^2}\tilde{\psi}$$

where we have used $\tilde{\zeta}_0 = \tilde{\nabla}^2\tilde{\psi}$, again coming from geostrophic balance.

Restoring the definitions of Fr and Ro,

$$\frac{\mathrm{Fr}^2}{\mathrm{Ro}^2} = \frac{U^2}{gH}\frac{f^2L^2}{U^2} = \frac{f^2L^2}{gH}.$$

Part of this parameter group involves a physical length scale

$$L_D \equiv \frac{\sqrt{gH}}{f}$$

called the "Rossby deformation length". The constants g, H and f are all physical — they are not characteristic values like L, D and U.

Hence, the PV can be re-written (again!) as

$$\tilde{q}_0 = \tilde{\nabla}^2 \tilde{\psi} - \frac{L^2}{L_D^2} \tilde{\psi}$$

This is very important. It means that, *given* the PV field $\tilde{q}_0$, one can find $\tilde{\psi}$ by solving a (linear) Helmholtz equation.

From shallow water to quasi-geostrophy

But there is more! Once $\tilde{\psi}$ is found, the velocity field is found by simple differentiation,

$$\tilde{u}_0 = -\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \qquad \text{and} \qquad \tilde{v}_0 = \frac{\partial \tilde{\psi}}{\partial \tilde{x}}$$

and this is all one needs to evolve the PV field:

$$\frac{\partial \tilde{q}_0}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}_0 \cdot \tilde{\boldsymbol{\nabla}} \tilde{q}_0 = 0.$$

This closes the system of equations. Only one scalar evolution equation is involved and moreover the velocity field is found by solving linear equations — something called "PV inversion".

The QG model is the simplest model in geophysical fluid dynamics.

From shallow water to quasi-geostrophy

Restoring dimensions, the characteristic scales L, D and U disappear:

$$q = \nabla^2\psi - \frac{\psi}{L_D^2}\,,$$

$$u = -\frac{\partial\psi}{\partial y} \qquad \text{and} \qquad v = \frac{\partial\psi}{\partial x}$$

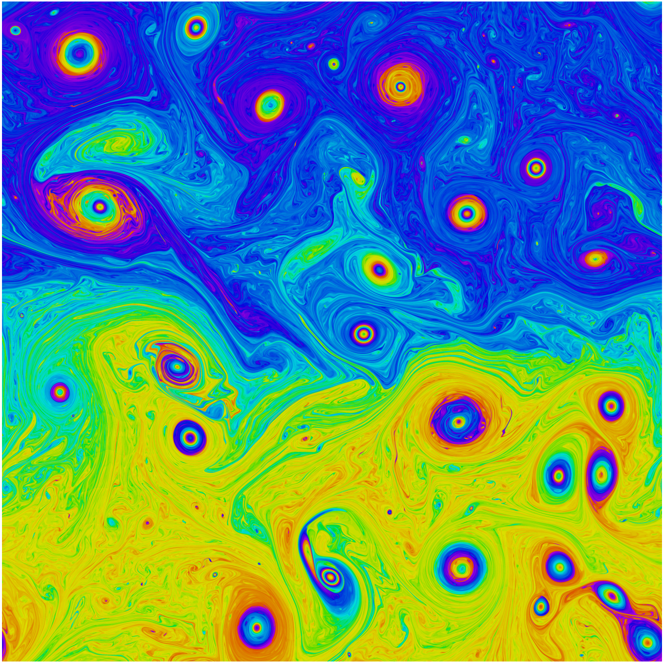
together with

$$\frac{\partial q}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} q = 0.$$

This is the complete, *dimensional*, quasi-geostrophic model.

It uses the basic ingredients in geophysical fluid dynamics in the simplest possible manner, leading to

- a single evolution equation for the *slow* material transport of PV, and
- linear *inversion relations* providing the flow field from the PV field.

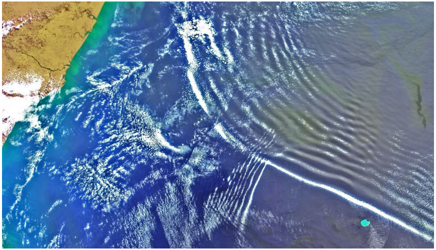


The process of reduction

The QG model is certainly capable of exhibiting complex turbulent flow evolution.

But what have we lost in the process of reduction? The "parent" shallow-water model has 3 time derivatives for 3 prognostic variables h, u and v. By contrast, the QG model has just one time derivative and one prognostic variable, the PV.

The QG model filters the relatively high frequency gravity waves, having frequencies $> f$. What is left is the low frequency evolution of PV.



The justification is that gravity waves tend to be energetically weak compared to the slow, ponderous evolution of PV.

From 3D to shallow water

We now step back and consider the origin of the shallow-water model itself. The "parent model" in this case is governed by the 3D Euler equations for constant density but with a free surface at $z = h(x, y, t)$.

The parent model is governed by the equations

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\boldsymbol{e}_z \times \boldsymbol{u} = -\nabla P, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial P}{\partial z} - g$$

$$\nabla \cdot \boldsymbol{u} + \frac{\partial w}{\partial z} = 0$$

where $P = p/\rho$ is a scaled pressure. Here $\boldsymbol{u} = (u, v)$ is the *horizontal* velocity and w is the *vertical* velocity. ∇ is the 2D gradient operator.

Boundary conditions:

$$w = 0, \qquad \frac{\partial P}{\partial z} = -g \qquad \text{at} \quad z = 0$$

$$w = \frac{\mathrm{D}h}{\mathrm{D}t}, \qquad P = \text{constant } (= 0) \quad \text{at} \quad z = h.$$

From 3D to shallow water

It is convenient to separate P into a *hydrostatic part*, $P_h = g(h - z)$, and a remaining *non-hydrostatic part*, $P_n = P - P_h$:

$$\Rightarrow \quad \frac{\partial P_n}{\partial z} = 0 \quad \text{at} \quad z = 0, \qquad P_n = 0 \quad \text{at} \quad z = h.$$

The momentum equations then become

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\boldsymbol{e}_z \times \boldsymbol{u} = -g\boldsymbol{\nabla} h - \boldsymbol{\nabla} P_n, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial P_n}{\partial z}.$$

Taking the 3D divergence of the momentum equations and using incompressibility provides a diagnostic equation for P_n:

$$\nabla^2 P_n + \frac{\partial^2 P_n}{\partial z^2} = f\zeta - g\nabla^2 h + 2J_{xy}(u, v) + 2J_{yz}(v, w) + 2J_{zx}(w, u)$$

where ∇^2 is the 2D Laplacian, $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vertical (relative) vorticity and J is the usual Jacobian operator.

From 3D to shallow water

When the mean depth H is small compared to a characteristic horizontal scale L, it is common to approximate the dynamics by the shallow-water (SW) equations or the Serre/Green-Naghdi (GN) equations.

Both the SW and GN models assume that $\boldsymbol{u}$ is independent of z.

$$\Rightarrow \quad w = -\delta z \quad \text{where} \quad \delta = \boldsymbol{\nabla} \cdot \boldsymbol{u}.$$

Applying this at $z = h$, we have

$$\frac{\mathrm{D}h}{\mathrm{D}t} = -\delta h \qquad \Rightarrow \qquad \frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot (h\boldsymbol{u}) = 0,$$

the usual mass-continuity equation.

If we now vertically average the horizontal momentum equations we obtain

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\boldsymbol{e}_z \times \boldsymbol{u} = -g\boldsymbol{\nabla} h - \frac{1}{h}\boldsymbol{\nabla}\bar{P}_n \qquad \text{where} \qquad \bar{P}_n \equiv \int_0^h P_n \mathrm{d}z$$

From 3D to shallow water

The SW model *furthermore* assumes $P_n = 0$ (hydrostatics), and this closes the system in the primitive variables h, u and v:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -g\frac{\partial h}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -g\frac{\partial h}{\partial y}, \qquad \frac{\partial h}{\partial t} + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0\,.$$

No asymptotic expansion in a small parameter is necessary!

Green & Naghdi (JFM, 1976) argue that such expansions do not guarantee consistency, i.e. conservation of energy, etc.

In the present context, they show that one can derive a consistent model *even more accurate than shallow-water* by *not* imposing the hydrostatic approximation.

This "GN model" also requires $\bar{P}_n$. Let's see how this is obtained.

From 3D to better than shallow water

By not imposing the hydrostatic approximation, we need to satisfy the complete vertical force balance:

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial P_n}{\partial z}.$$

But incompressibility gives $w = -\delta z$, where $\delta = \boldsymbol{\nabla} \cdot \boldsymbol{u}$ is the horizontal divergence. Since δ is independent of z, the above equation becomes

$$-\frac{\mathrm{D}\delta}{\mathrm{D}t} z - \delta w = -\left(\frac{\mathrm{D}\delta}{\mathrm{D}t} - \delta^2\right) z = -\frac{\partial P_n}{\partial z}$$

Now we simply integrate w.r.t. z to find

$$P_n = P_0(x, y, t) + \frac{1}{2}\left(\frac{\mathrm{D}\delta}{\mathrm{D}t} - \delta^2\right) z^2$$

where P_0 is determined from the boundary conditions satisfied by P_n.

From 3D to better than shallow water

At $z = 0$, $\partial P_n / \partial z = 0$ is automatically satisfied.

At $z = h$, the boundary condition is $P_n = 0$, and this determines P_0:

$$P_0 = -\frac{1}{2}\left(\frac{\mathrm{D}\delta}{\mathrm{D}t} - \delta^2\right) h^2 \qquad \text{so} \qquad P_n = P_0\left(1 - \frac{z^2}{h^2}\right).$$

For the reduced model, we need the layer-integrated non-hydrostatic pressure,

$$\bar{P}_n = \int_0^h P_n \mathrm{d}z = \frac{2}{3} P_0 h \qquad \Rightarrow \qquad \bar{P}_n = -\frac{1}{3}\left(\frac{\mathrm{D}\delta}{\mathrm{D}t} - \delta^2\right) h^3$$

In this form, the GN equations are *implicit*, since time derivatives appear on both sides of the momentum equations.

But an explicit model can be obtained from the divergence of the horizontal momentum equations,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\boldsymbol{e}_z \times \boldsymbol{u} = -g\boldsymbol{\nabla} h - \frac{1}{h}\boldsymbol{\nabla}\bar{P}_n\,.$$

applying $\boldsymbol{\nabla}\cdot$ and a number of vector calculus identities, one finds

$$\frac{\mathrm{D}\delta}{\mathrm{D}t} - \delta^2 = \tilde{\gamma} - \boldsymbol{\nabla}\cdot\left(\frac{1}{h}\boldsymbol{\nabla}\bar{P}_n\right)$$

where

$$\tilde{\gamma} \equiv f\zeta - g\nabla^2 h + 2J_{xy}(u,v) - 2\delta^2$$

just depends on h, u and v — *not their time derivatives*.

From 3D to better than shallow water

But previously we found, from the vertical force balance,

$$\frac{\mathrm{D}\delta}{\mathrm{D}t} - \delta^2 = -\frac{3}{h^3}\bar{P}_n$$

(after a little re-arrangement).

Thus, upon eliminating $\mathrm{D}\delta/\mathrm{D}t - \delta^2$, we obtain

$$\nabla \cdot \left(\frac{1}{h}\nabla\bar{P}_n\right) - \frac{3}{h^3}\bar{P}_n = \tilde{\gamma},$$

an *explicit*, *linear*, *elliptic* equation to determine $\bar{P}_n$ from $\tilde{\gamma}$ and h.

This "Vertically-Averaged" (VA) model (D & Jalali, JFM A33, 2020) has a similar form to the shallow-water (SW) model, only now with the addition of non-hydrostatic pressure $\bar{P}_n$, a quantity *diagnosed* from h, u and v at each instant of time.

From 3D to better than shallow water

There are three prognostic equations: two for momentum

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -g\frac{\partial h}{\partial x} - \frac{1}{h}\frac{\partial \bar{P}_n}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -g\frac{\partial h}{\partial y} - \frac{1}{h}\frac{\partial \bar{P}_n}{\partial y}$$

and one for mass

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0\,,$$

together with a diagnostic equation for "pressure":

$$\boldsymbol{\nabla}\cdot\left(\frac{1}{h}\boldsymbol{\nabla}\bar{P}_n\right) - \frac{3}{h^3}\bar{P}_n = \tilde{\gamma} \equiv f\zeta - g\nabla^2 h + 2J_{xy}(u,v) - 2\delta^2\,.$$

Moreover, and unlike in the SW model, all conservation laws simply follow from vertically-averaging those in the parent 3D model!

From 3D to better than shallow water

This demonstrates consistency: all conservation laws are retained (see also the variational formulation of Miles & Salmon, JFM 1985).

For example, total (kinetic plus potential) energy is

$$E = \frac{1}{2} \iint h \left(u^2 + v^2 + \frac{1}{3} h^2 \delta^2 + gh \right)$$

where the red term comes from integrating w^2 over z. *It must be excluded in the SW model.*

Importantly, a material invariant of the 3D parent model, namely potential vorticity, is also a material invariant of the reduced 2D model upon vertical averaging:

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \qquad \text{where} \qquad q = \frac{\zeta + f}{h} + \frac{1}{3} J_{xy}(\delta, h)$$

The red term must be excluded in the SW model.

Potential vorticity in the VA model

True to its name, in the VA model the potential vorticity $q(x, y, t)$ is the *vertically-averaged* Beltrami-Rossby-Ertel potential vorticity $Q(x, y, z, t)$:

$$Q = \boldsymbol{\omega}_a \cdot \boldsymbol{\nabla}\theta \qquad ; \qquad \boldsymbol{\omega}_a = \boldsymbol{\omega} + f\,\boldsymbol{e}_z$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$, and θ is a materially-conserved field $(\mathrm{D}\theta/\mathrm{D}t = 0)$ representing a coordinate surface ($\theta = 0$ on $z = 0$, and $\theta = 1$ on the free surface $z = h$). Then

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0$$

in the full 3D Euler equations (Beltrami, 1871).

Let $z = \mathfrak{z}(x, y, \theta, t) = \theta h$ be the height of the surface $\theta =$ constant. Then

$$\theta = \frac{z}{h} \qquad \text{and} \qquad \boldsymbol{\nabla}\theta = \frac{\boldsymbol{e}_z}{h} - z\frac{\boldsymbol{\nabla} h}{h^2}.$$

Potential vorticity in the VA model

Also, since u and v are independent of z,

$$\boldsymbol{\omega} = \left(\frac{\partial w}{\partial y} \,, \; -\frac{\partial w}{\partial x} \,, \; \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \left(-z \frac{\partial \delta}{\partial y} \,, \; z \frac{\partial \delta}{\partial x} \,, \; \zeta \right)$$

using $w = -\delta z$ (incompressibility).

Forming $\boldsymbol{\omega}_a \cdot \boldsymbol{\nabla}\theta$, we find

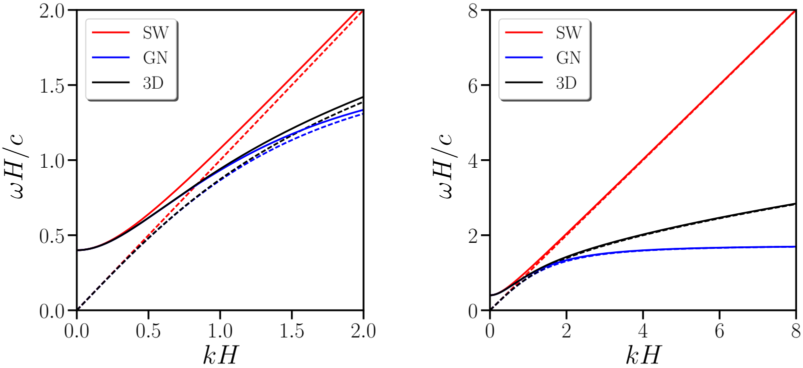
$$Q = \frac{\zeta + f}{h} + \frac{z^2}{h^2} J_{xy}(\delta, h).$$

Vertically averaging, we obtain the Serre-GN (or VA) potential vorticity:

$$q = \frac{\zeta + f}{h} + \frac{1}{3} J_{xy}(\delta, h).$$

What are the advantages of the VA (GN) model over the SW model?

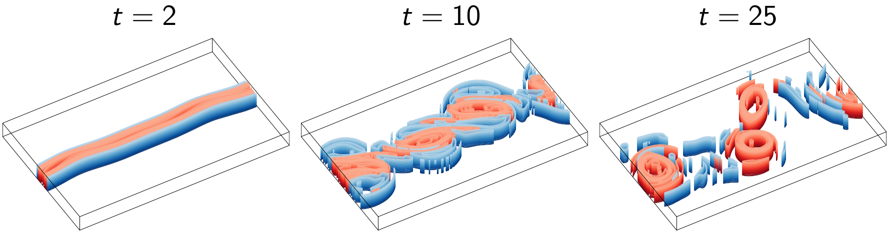
First of all, gravity wave frequencies ω are more accurately captured:



Dashed: $f = 0$ (non-rotating); Solid: $f = 4\pi$ (rotating).

Here, k is wavenumber and we have taken $c = \sqrt{gH} = 2\pi$ and $H = 0.2$.

Evolution of $\zeta - f\tilde{h}$ in 3D Euler with a free surface



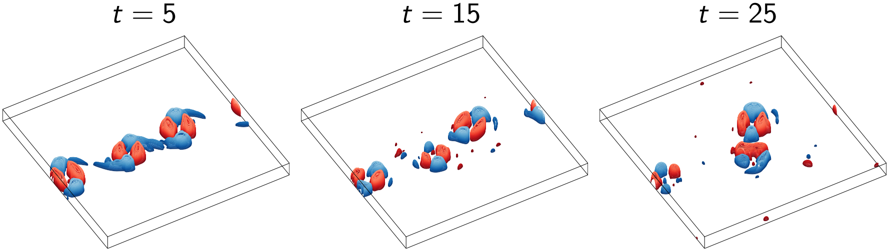
This field remains remarkably 2D, consistent with the SW and GN model assumptions. (The images show $|x| \leq \pi$ and $y \leq 2$ for all θ.)

Here and below, the mean depth $H = 0.4$.

The Rossby number $\zeta_{\max}/f$ varies from 0.57 to 0.71.

The Froude number $(\|\boldsymbol{u}\|/\sqrt{gh})_{\max}$ varies from 0.17 to 0.23.

Flow evolution in 3D — non-hydrostatic pressure P_n

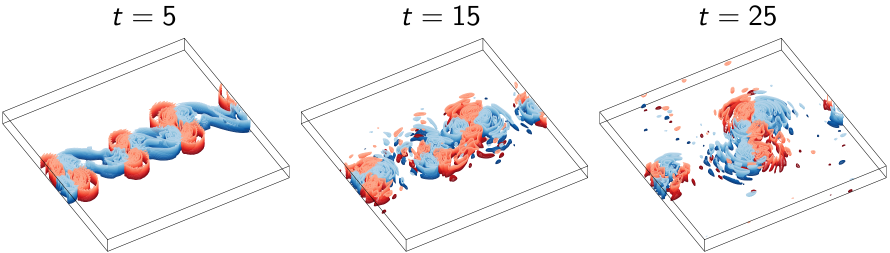


This field varies strongly in z (the full domain is shown). But this is consistent with the boundary conditions

$$\frac{\partial P_n}{\partial z} = 0 \quad \text{at} \quad z = 0, \qquad P_n = 0 \quad \text{at} \quad z = h.$$

One expects a quadratic dependence in z, consistent with the paraboloidal structures seen.

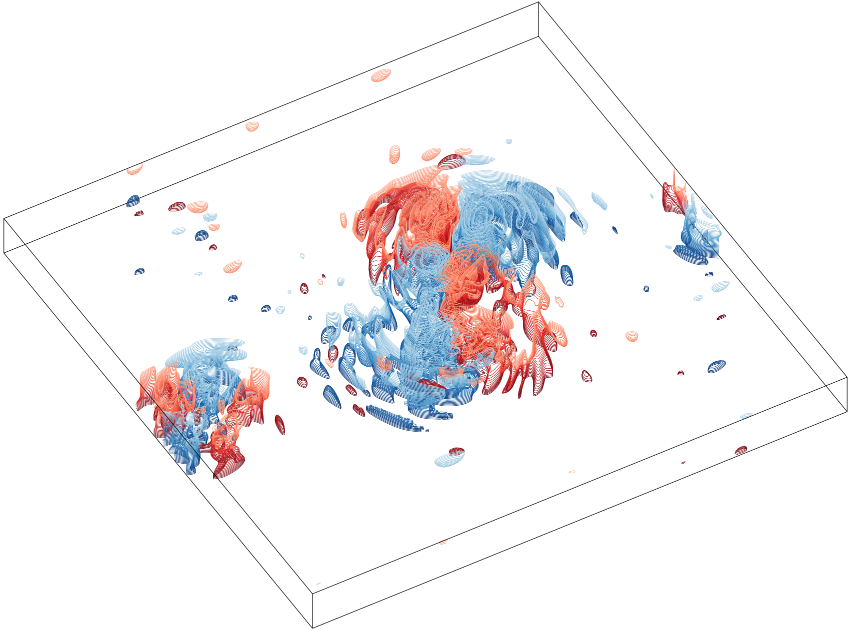
Flow evolution in 3D — horizontal divergence δ



This field varies strongly in z. This is *not consistent* with the SW and GN model assumptions.

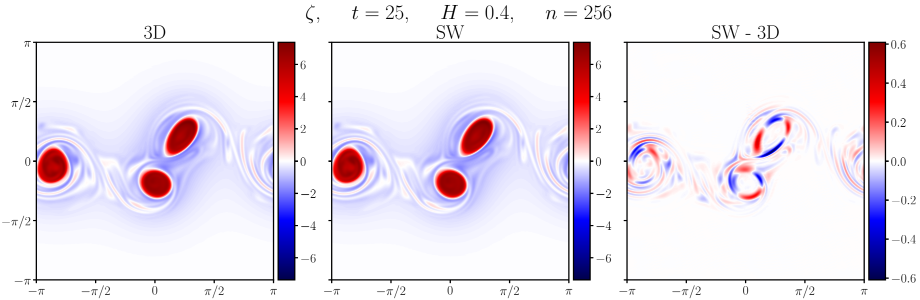
At $t = 0$, δ (like $\boldsymbol{u}$) is independent of z.

Prominent 3D variations develop in time.



Comparisons with the 3D model

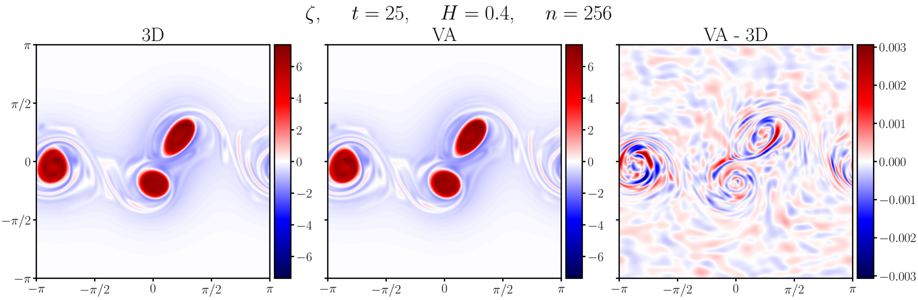
A second advantage of the VA model is that the vortical flow ζ is much more accurately captured:



Here we compare the vertically-averaged 3D vertical vorticity ζ at a late time t with that in the SW model. Maximum errors are around 10%.

Comparisons with the 3D model

Again, but now comparing the 3D and VA models:



Differences are 200 times smaller! There is a *substantial* gain in accuracy by including non-hydrostatic effects.

Summary

We have focused on models in geophysical fluid dynamics, relevant to research in atmospheric and oceanic science.

- We have showed how to developed "reduced" models from more complete "parent" models.
- Such models are important research tools, even if they cannot forecast the weather!
- We showed how to derive the quasi-geostrophic model from the parent shallow-water one using an asymptotic expansion in a small parameter (the Rossby number).
- We also described an entirely different approach, making an ansatz (avoiding an asymptotic expansion), to derive (an improved) shallow-water model from the parent 3D Euler equations.

Next: Focus on reduced models.