Fast and Slow: The Dynamics of Superrotation Phenomena in Planetary Atmospheres: II. Simple models & theories

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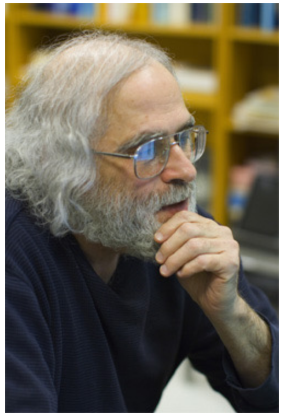
Plan

- Simple model of Hadley circulation (Held & Hou)
- Super-rotation with meridional overturning and diffusive eddies (Gierasch)
- Extension with strong momentum diffusion (Yamamoto et al.)
- Angular momentum mixing
- Classification of waves
- Wave-zonal flow interactions and non-acceleration

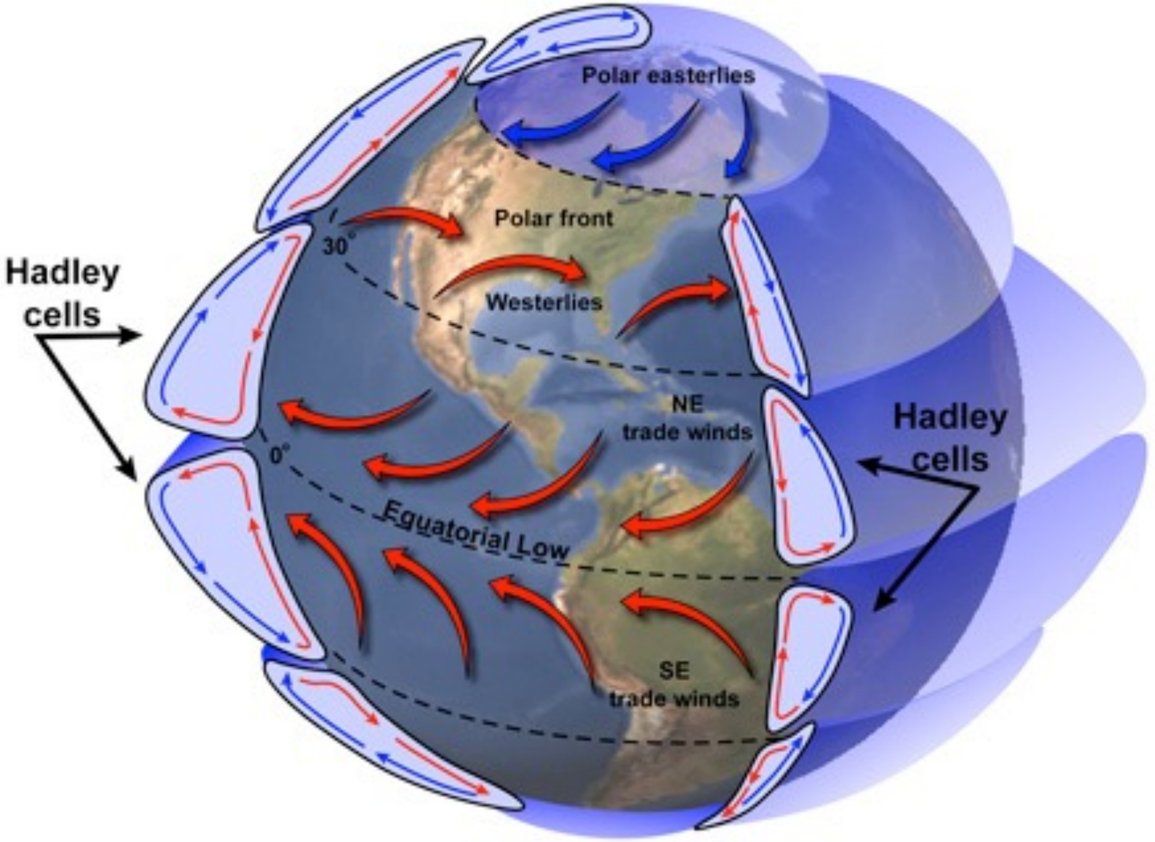
Simple theoretical models: Held & Hou (1980)

- Aim is to determine properties and width of axisymmetric Hadley circulation and associated zonal flow and thermal structure
  - Consistent with steady thermal balance
  - Hydrostatic and gradient wind balance



Isaac Held

Arthur Y. Hou

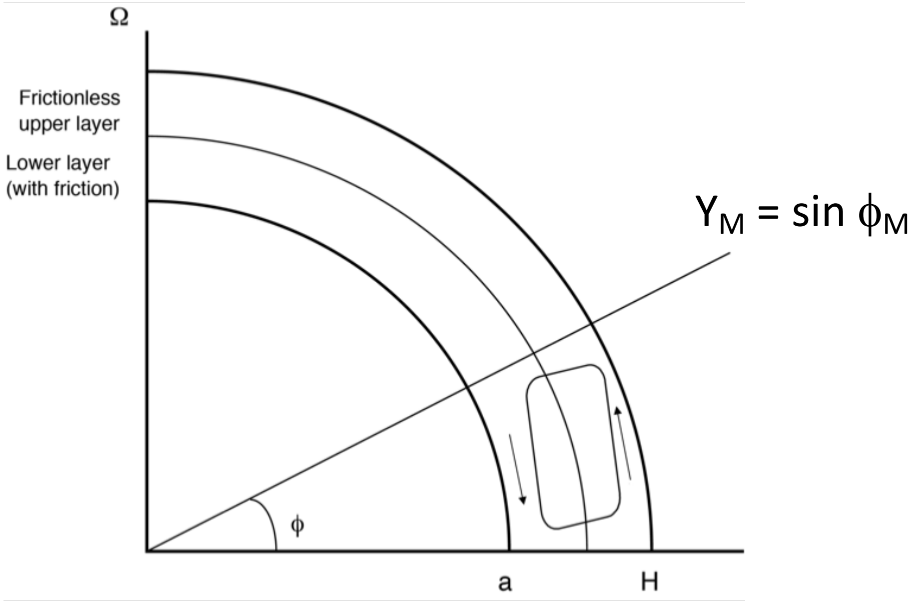


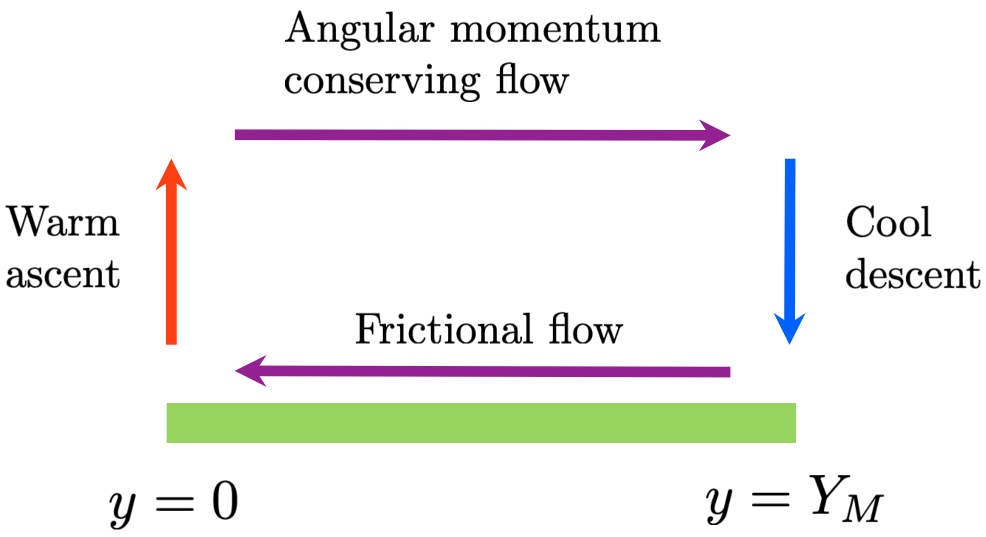
Simple theoretical models: Held & Hou (1980)

- Two-layer atmosphere
  - Quasi-inviscid so angular momentum conservation applies in the free atmosphere by the overturning circulation
  - Hydrostatic and Gradient wind balance at all $y = \sin\varphi$

$$\frac{\partial}{\partial z}\left(fu + \frac{u^2 \tan\varphi}{a}\right) = -\frac{g}{a\theta_0}\frac{\partial\theta}{\partial\varphi}$$

  - Weak flow ($u \approx 0$) near the surface due to frictional drag





Simple theoretical models: Held & Hou (1980)

- Asssume a radiative equilibrium potential temperature profile

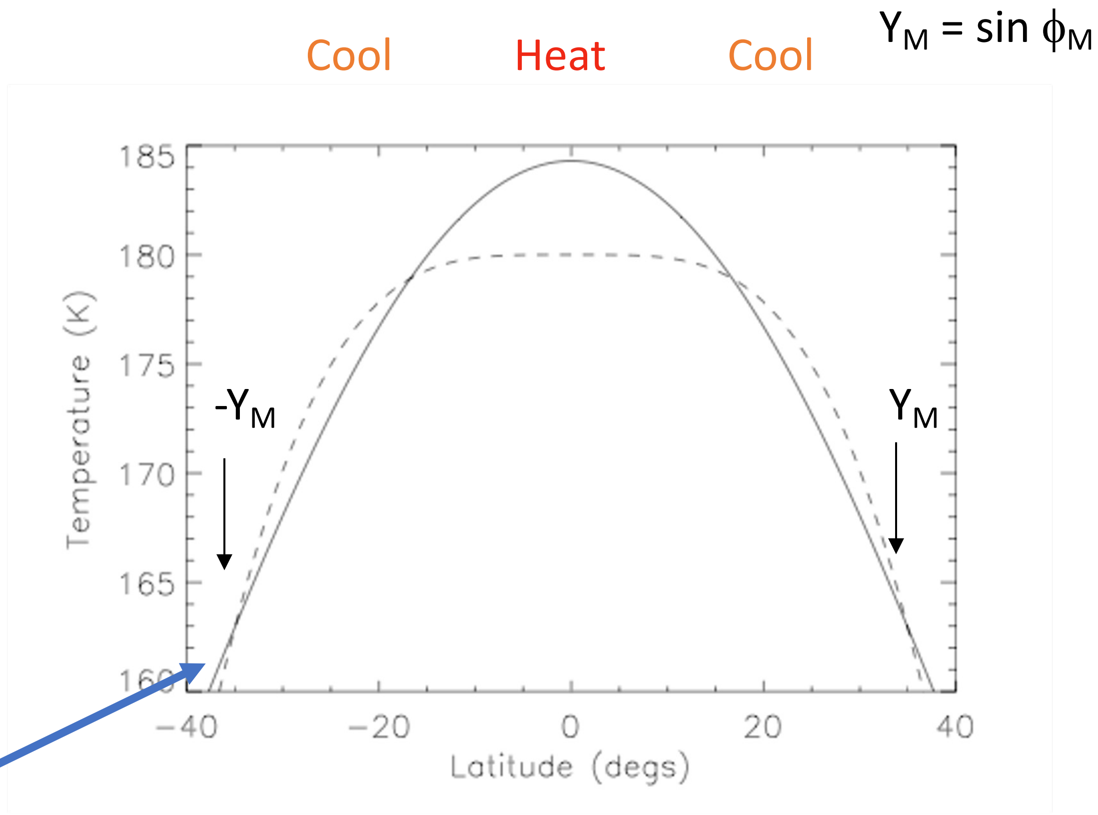
$$\frac{\theta_{eq}}{\theta_o} = 1 - \frac{2}{3}\Delta_h P_2(\sin\varphi) + \Delta_v\left(\frac{z}{H} - \frac{1}{2}\right)$$

  - $\theta_0\Delta_h$ is equator-pole potential temperature contrast
  - $\theta_0\Delta_v$ is vertical potential temperature contrast over *z=0-H*
  - $P_2(y) = \frac{1}{2}(3y^2 - 1)$

- Take integrated vertical average of () and ()

$$fu + \frac{u^2\tan\varphi}{a} = -\frac{gH}{a\theta_0}\frac{\partial\bar{\theta}}{\partial\varphi}$$

$$\bar{\theta}_{eq} = \theta_0\left[1 - \frac{2}{3}\Delta_h P_2(\sin\varphi)\right]$$



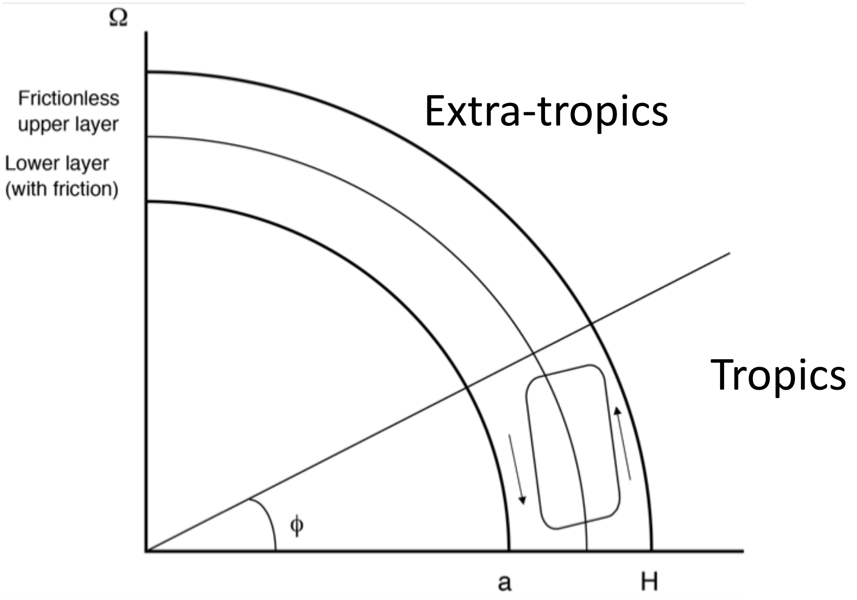
Simple theoretical models: Held & Hou (1980)

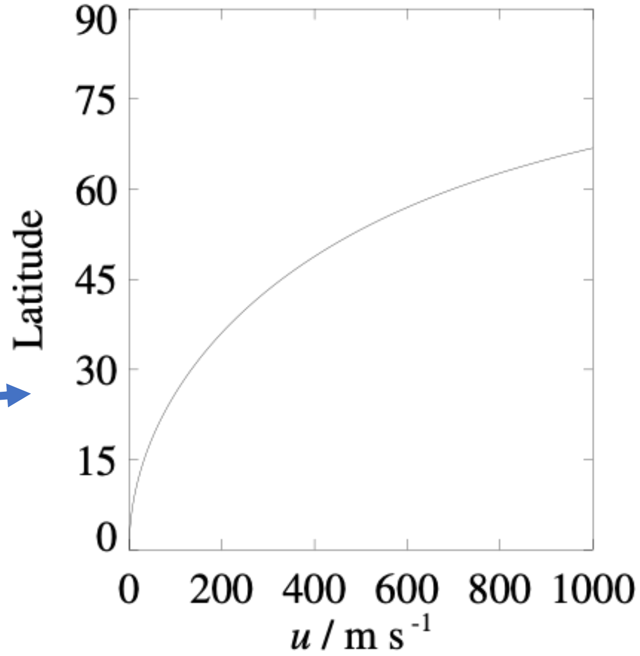
- Assume meridional velocity $v = 0$ in the ***extra-tropical*** free atmosphere.
- From gradient thermal wind balance

$$u_{ET} = \Omega a \cos\varphi \left(\sqrt{2\mathcal{R}\frac{Z}{H} + 1} - 1 \right)$$

  - Where $\mathcal{R} = \frac{\Delta_h g H}{\Omega^2 a^2}$ is thermal Rossby number
- Within the Hadley cell *u* is determined by AM conservation, assuming *u = 0* on the equator.

$$u_{HC} = \frac{\Omega a \sin^2\varphi}{\cos\varphi}$$





Simple theoretical models: Held & Hou (1980)

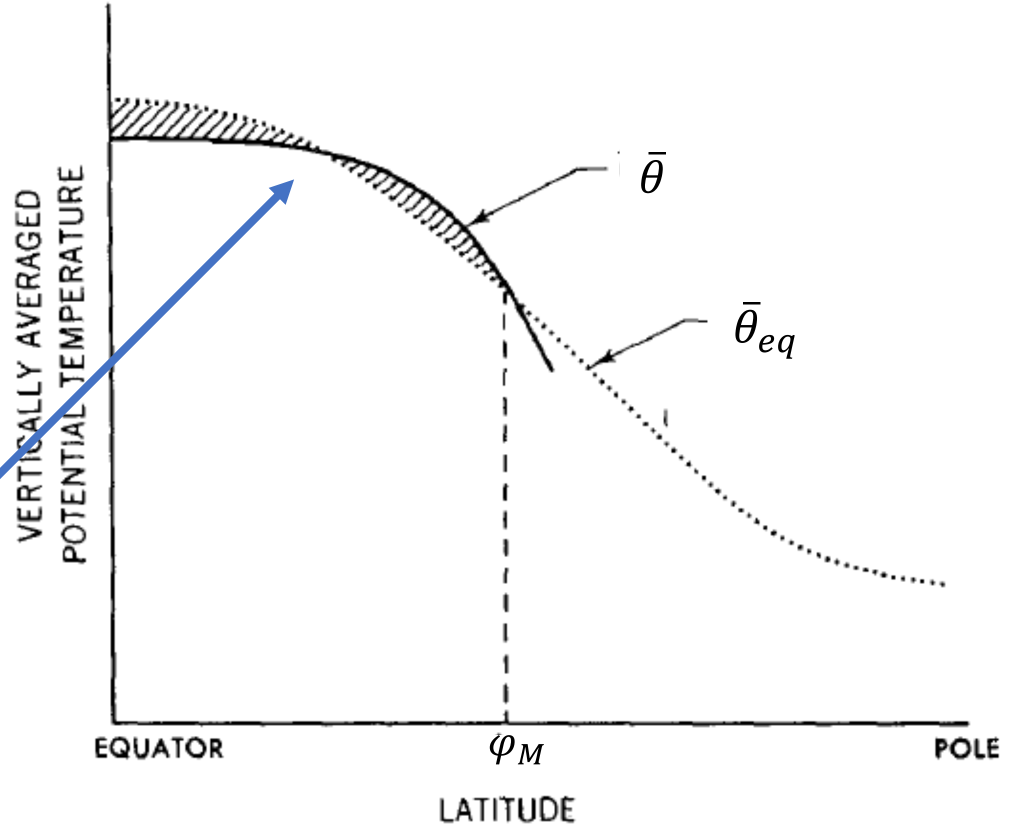
- Use gradient wind balance to infer the vertical mean temperature, so in the Hadley cell
  - $\frac{\bar{\theta}(0)-\bar{\theta}}{\theta_0} = \frac{u_{HC}^2}{2gH}$ for $|y|<Y_M$
- Solve for Y_M or φ_M using matching conditions

$$\bar{\theta}(\varphi_M^+) = \bar{\theta}(\varphi_M^-)$$

  - And the thermal balance condition for the Hadley cell

$$\int_0^{\varphi_M} \bar{\theta} \cos\varphi \, d\varphi = \int_0^{\varphi_M} \bar{\theta}_{eq} \cos\varphi \, d\varphi$$

$Y_M = \sin \phi_M$



Simple theoretical models: Held & Hou (1980)

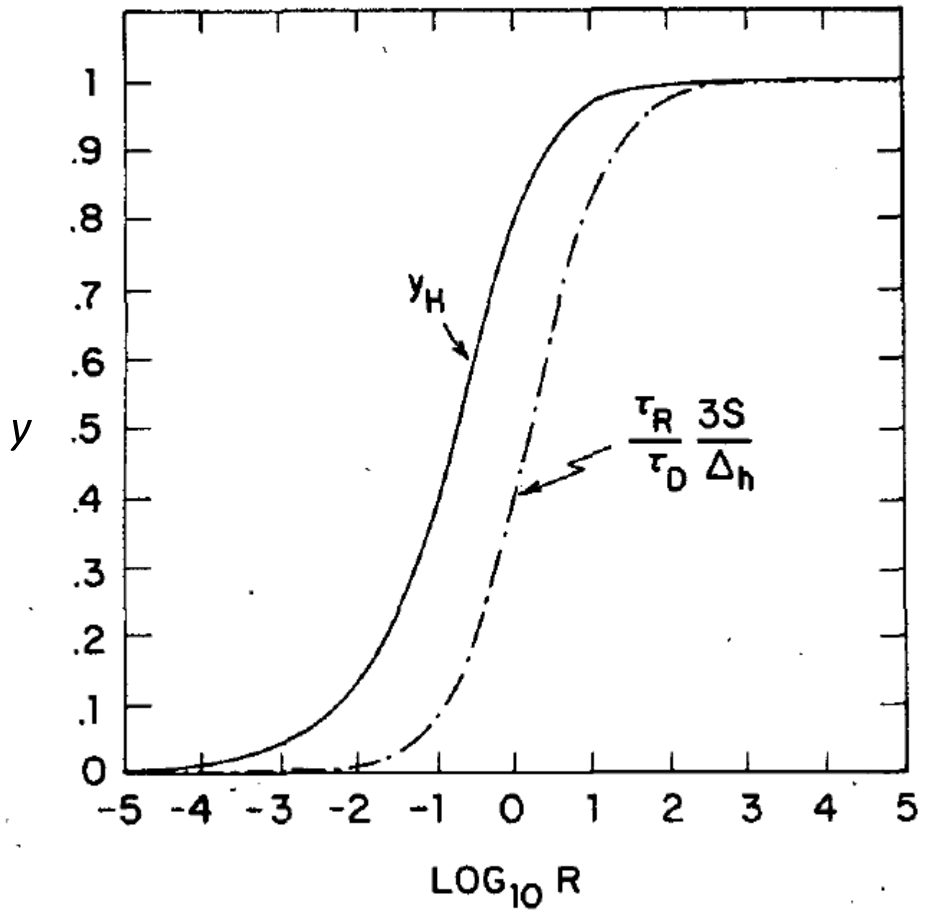
- Solution for y_M can be expressed in terms of the root of the equation

$$\frac{1}{3}(4\mathcal{R}-1)y_M^3 - \frac{y_M^5}{1-y_M^2} - y_M + \frac{1}{2}\ln\left[\frac{1+y_M}{1-y_M}\right] = 0$$

  - Or rearranging

$$\mathcal{R} = \frac{3}{4}\left[\frac{1}{3} + \frac{1}{y_M^2} + \frac{y_M^2}{1-y_M^2} - \frac{1}{2y_M^3}\ln\left[\frac{1+y_M}{1-y_M}\right]\right]$$

  - Which can be solved numerically



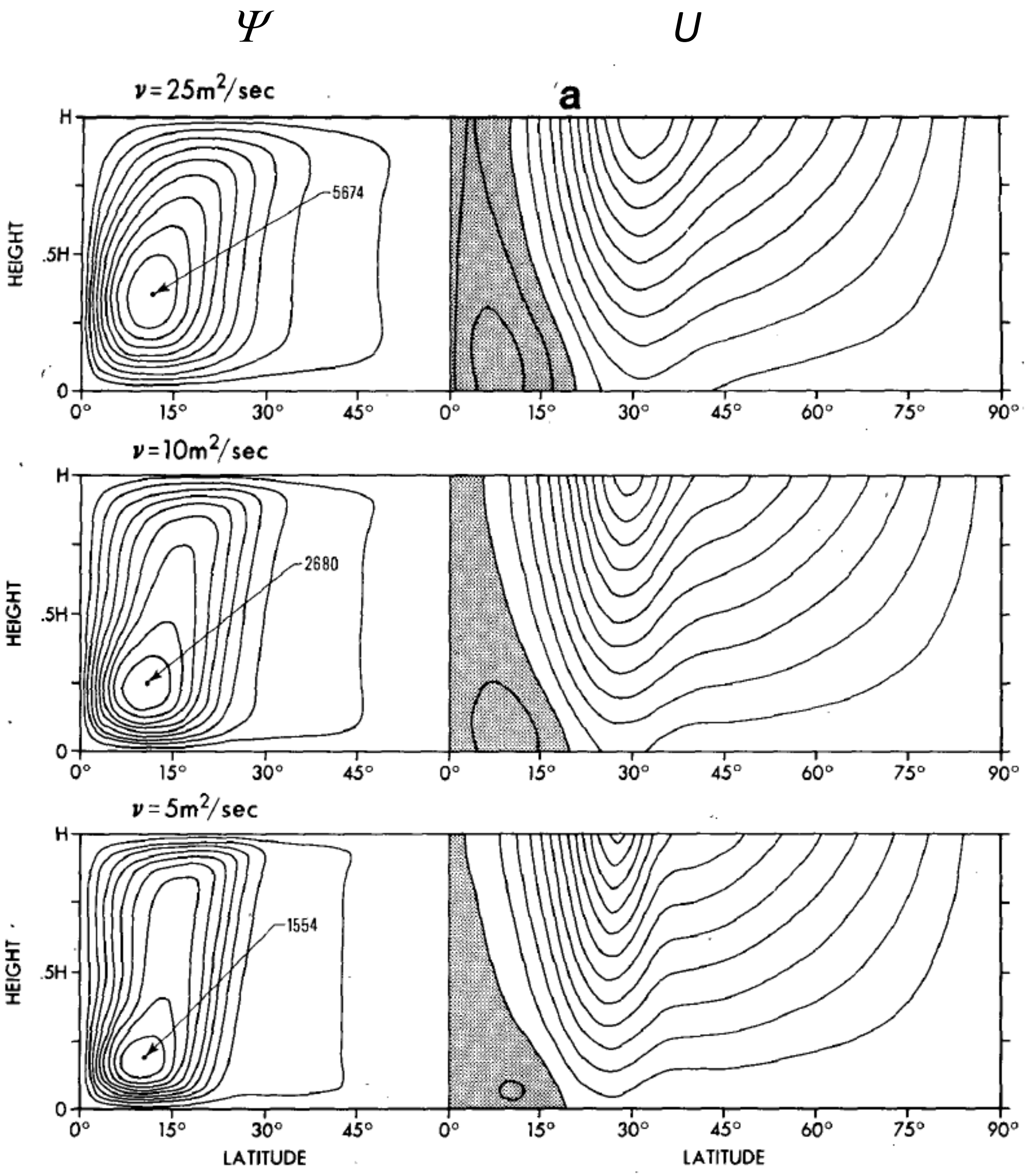
- Asymptotic limits can be found for
  - $\mathcal{R} \ll 1$: $y_M \approx \left(\frac{5\mathcal{R}}{3}\right)^{1/2}$
  - $\mathcal{R} \gg 1$: $y_M \approx 1 - \frac{3}{8\mathcal{R}}$

Simple theoretical models: Held & Hou (1980)

- Verify for continuously stratified, weakly viscous fluid (ν in the vertical **only**)
  - Note that VERTICAL viscosity can only diffuse *m* DOWNGRADIENT
  - Hence there can be no local maximum of *m* exceeding Ωa^2
- Solve time-dependent, 2D, axisymmetric primitive equations numerically with heating/cooling via Newtonian relaxation

$$\frac{D\theta}{Dt} = \frac{\theta_R - \theta}{\tau}$$

- Meridional streamfunction width consistent with inviscid y_M estimate
- Strength of circulation $\sim 1/\nu$
- $U \leq 0$ on/near the equator
- U ~ AM-conserving for $\varphi \leq \varphi_M$
- U varies smoothly $\rightarrow$ 0 from $\varphi_M \rightarrow 90^o$



Simple theoretical models: Held & Hou (1980)

- Estimate S_{global} for axisymmetric circulations by using solutions for u_{HC} and u_{ET} as given above.
  - $u = u_{HC}$ for $0 \leq \phi \lesssim \phi_M$ [$p \leq 0.8p_S$]
  - $u = u_{ET}$ for $\phi_M < \phi \leq \pi/2$ [for all p]

$$S = \frac{\int \rho a u \cos\varphi \, dV}{\int \rho \Omega^2 a^2 \cos^2\varphi \, dV} = S_{HC} + S_{ET}$$

where

$$S_{HC} = \frac{\int_0^{0.8p_S} \int_0^{\varphi_M} u_{HC} \cos^2\varphi \, d\varphi \, dp/g}{\int_0^{0.8p_S} \int_0^{\pi/2} \Omega a \cos^3\varphi \, d\varphi \, dp/g} \; ; \quad S_{ET} = \frac{\int_0^{p_S} \int_{\varphi_M}^{\pi/2} u_{ET} \cos^2\varphi \, d\varphi \, dp/g}{\int_0^{p_S} \int_0^{\pi/2} \Omega a \cos^3\varphi \, d\varphi \, dp/g}$$

[Lewis et al. 2021]

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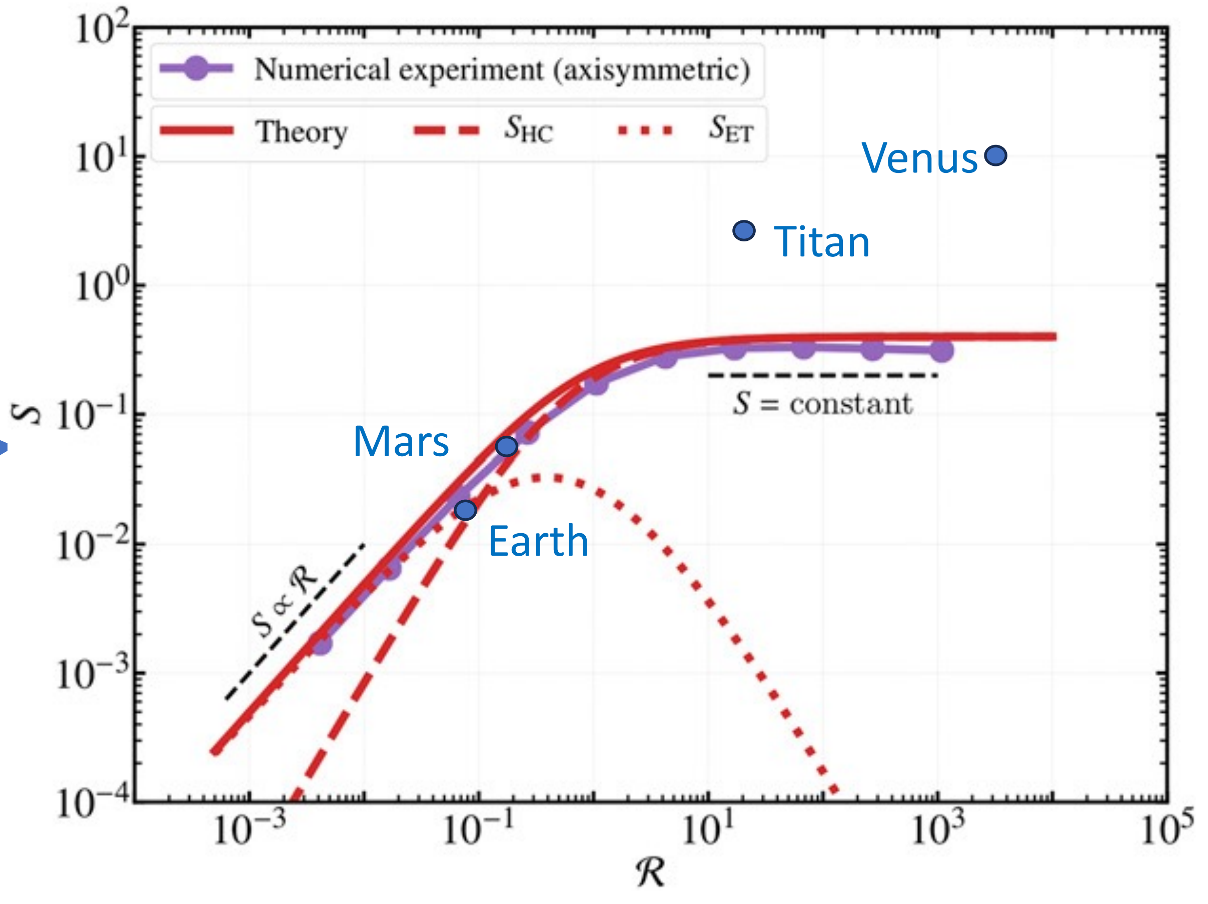
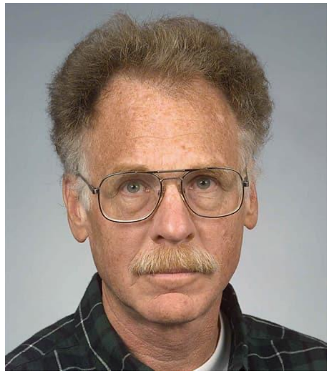
  - $u = u_{HC}$ for $0 \leq \phi \leq \phi_M$ [$p \leq 0.8p_s$]
  - $u = u_{ET}$ for $\phi_M < \phi \leq \pi/2$ [for all p]
- S depends primarily on $\mathcal{R}$
- Asymptotic limits can be found for
  - $\mathcal{R} \ll 1$: $S \approx \frac{1}{2}\mathcal{R}$
  - $\mathcal{R} \gg 1$: $S \approx \frac{2}{5} = \text{constant}$



[Lewis et al. 2021]

Simple theoretical models: Gierasch (1975)



Peter Gierasch

- Conceptual model to maintain a strong quasi-axisymmetric super-rotation
- Angular momentum is transferred upwards and polewards by a global scale meridional overturning circulation
- Eddies transfer AM equatorwards (up-gradient), accelerating super-rotation in the tropics
  - Parameterised as an anisotropic viscosity (ν_H, ν_V) with $\nu_H \gg \nu_V$
- Diffusion of heat is neglected
  - Otherwise it erodes the equator-pole temperature contrast
  - Reduces upper level U which should be in cyclostrophic balance for fast super-rotation

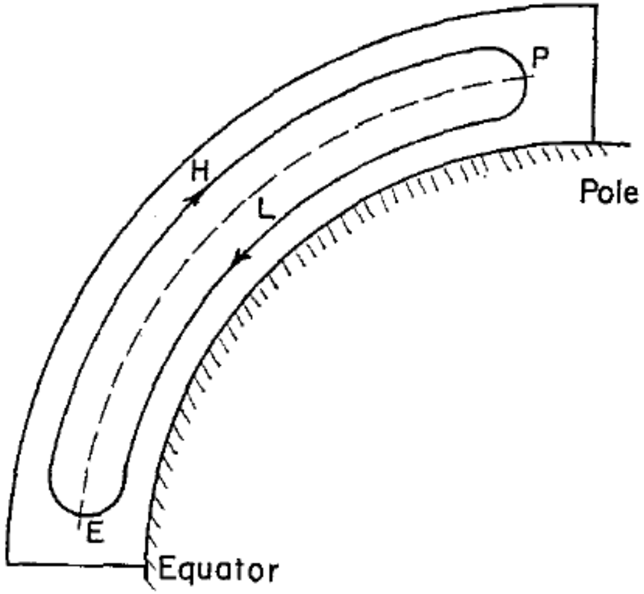
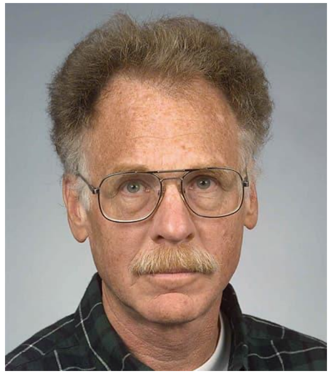


FIG. 1. Schematic of meridional cell. If the angular momentum per unit mass is greater at point E than at P, the meridional cell produces an upward flux of angular momentum. If it is greater at point H that at L, there is a poleward transport.

Simple theoretical models: Gierasch (1975)



Peter Gierasch

- Scale analysis and steady state solution of axisymmetric primitive equations, assuming simple spatial structures of key variables
  - Eddy thermal diffusivity assumed small enough to be negligible

$$1 \ll \frac{K_H H}{W a^2} \ll Ri = \frac{N^2}{dU/dz}$$

so temperature structure largely determined by radiative equilibri and $W \sim Q_R/N^2$

  - Strong super-rotation maintained by horizontal viscosity provide

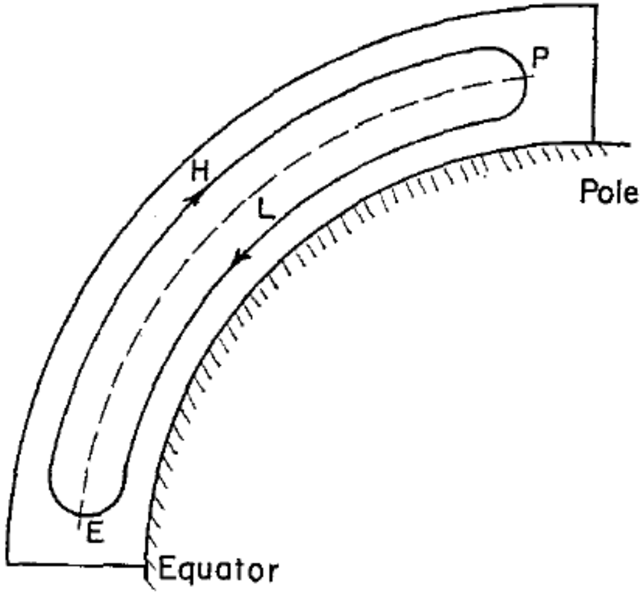
$$\tau_H (= \frac{a^2}{\nu_H}) \ll \tau_{HC} (= \frac{H}{W}) \ll \tau_V (= \frac{H^2}{\nu_V})$$

  - Cyclostrophic/gradient wind balance prevails if

$$\frac{\nu_V}{WH} \ll 1 \ll \frac{\nu_H H}{W a^2} \ll \left(\frac{UH}{Wa}\right)^2$$

- If these conditions are satisfied then

$$S \sim \frac{U}{\Omega a} \sim \exp\left(\frac{WH}{\nu_V}\right) > 1$$



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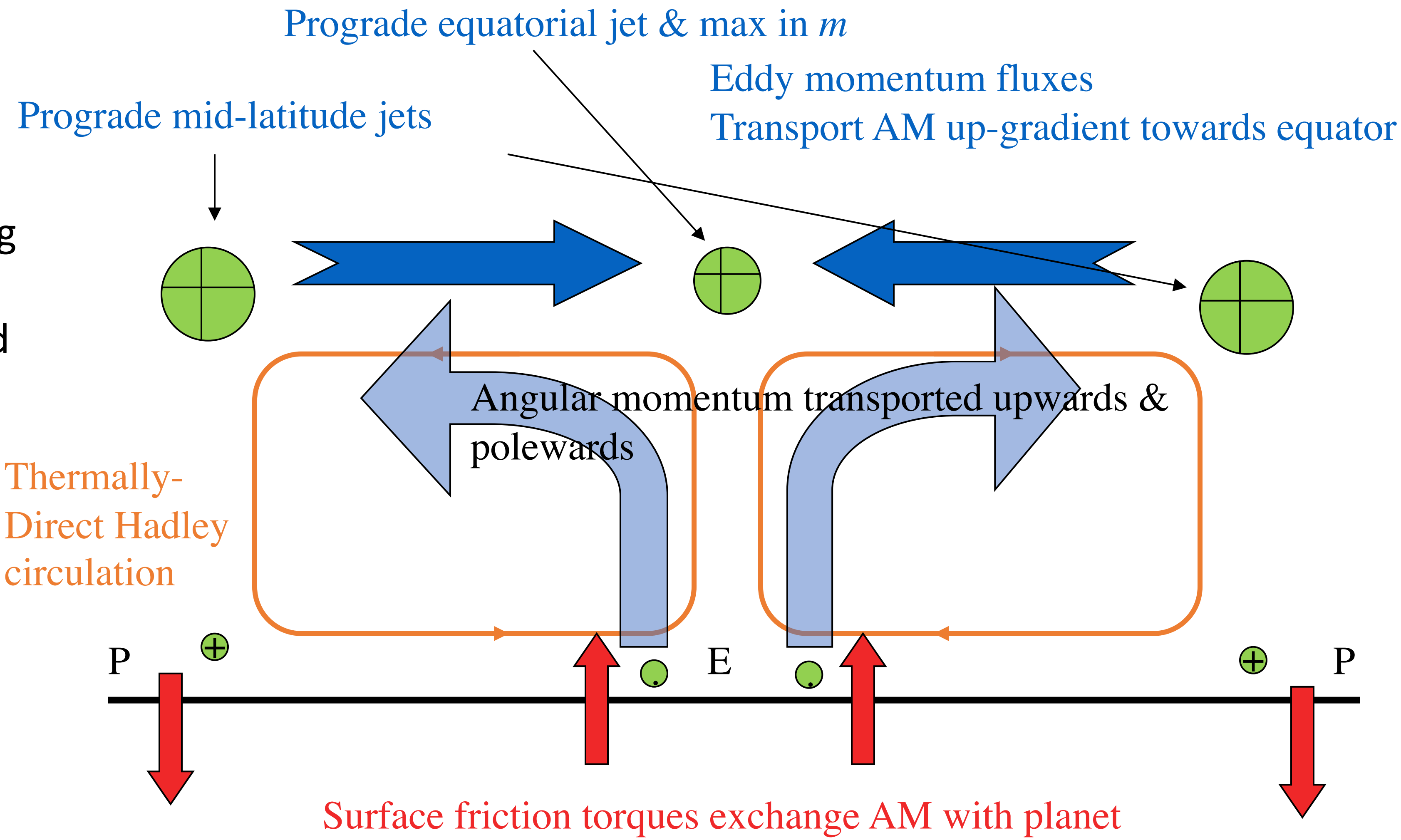
Gierasch/Rossow-Williams conceptual model Summary

- Super-rotation driven/maintained by combination of
  - axisymmetric overturning circulation and
  - up-gradient equatorward eddy transport of AM
- Rossow & Williams (1979)
  - generation and dissipation of barotropic Rossby waves act as AM transport mechanism via barotropic instability?



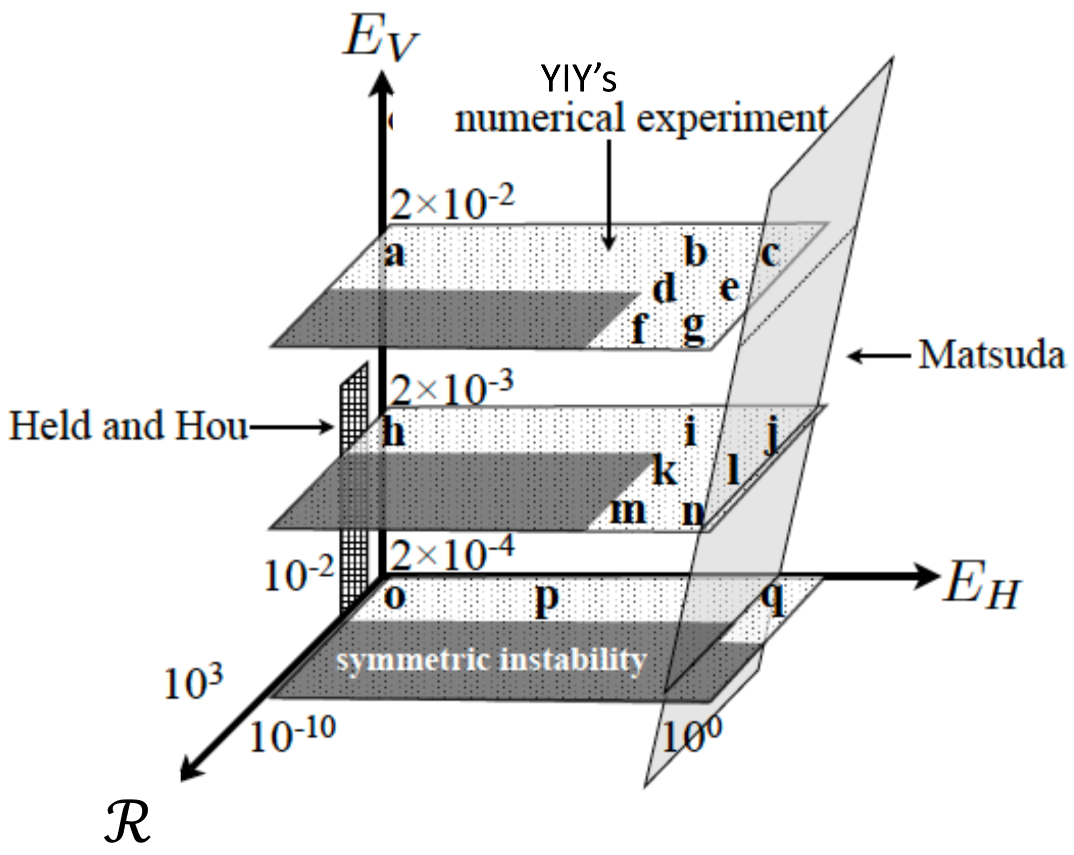
Simple theoretical/numerical models: Parameter sweeps

Peter Gierasch

- Exploration of a 3D dimensionless parameter space using time-dependent, 2D axisymmetric numerical models with Newtonian heating/cooling (cf Held & Hou 1980), solving primitive equations on a sphere
  - Matsuda (1980; 1982)
  - Yamamoto et al. (2009 - YIY);
  - Yamamoto & Yoden (2013 - YY)
- Define key external parameters to vary
  - Thermal Rossby number $\mathcal{R} = \frac{\Delta_h g H}{\Omega^2 a^2}$ (as before)
  - Horizontal and vertical Ekman numbers

$$E_V = \frac{\nu_V}{\Omega H^2}; E_H = \frac{\nu_H}{\Omega a^2}$$

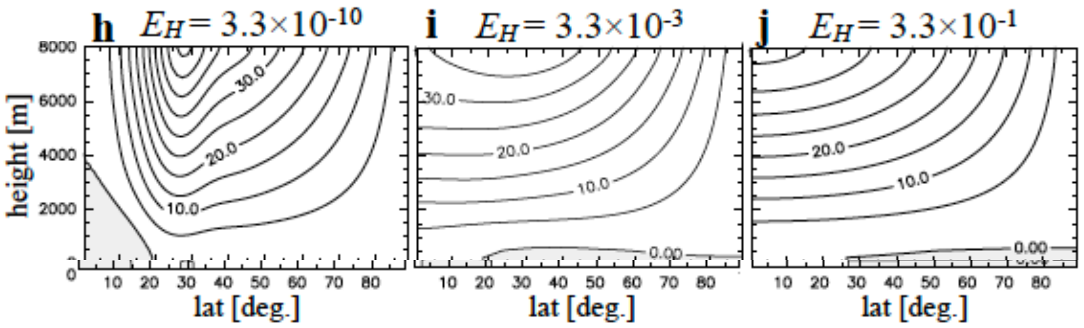
  - Other dimensionless groups fixed



Simple theoretical/numerical models: Parameter sweeps

$E_V = 2.1\times10^{-3}$

$\mathcal{R} = 0.12$



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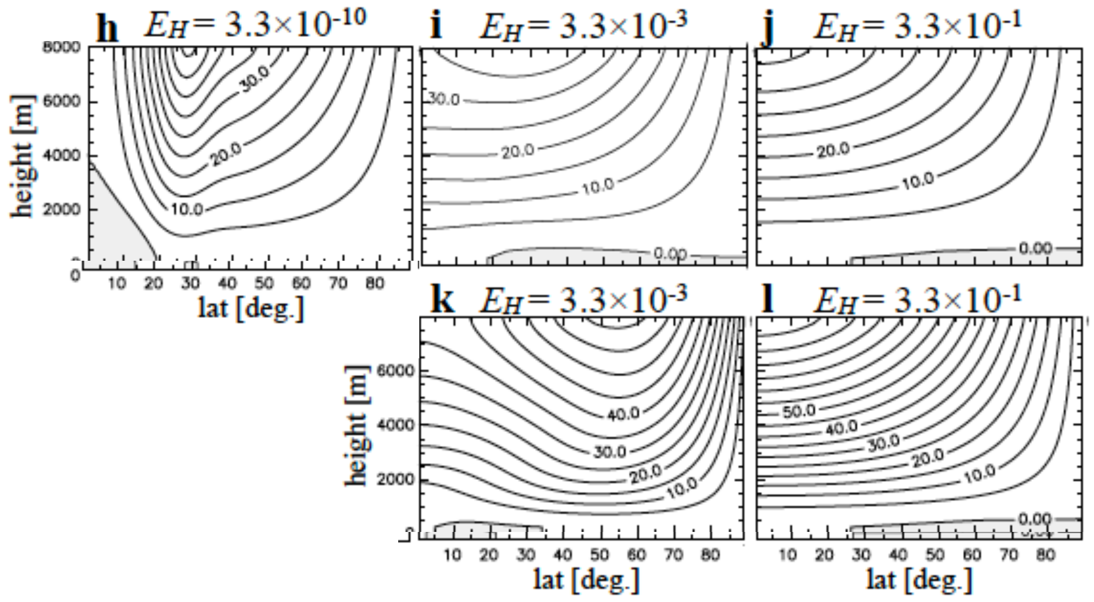
$E_V = 2.1\times10^{-3}$

$\mathcal{R} = 0.12$

$\mathcal{R} = 12$



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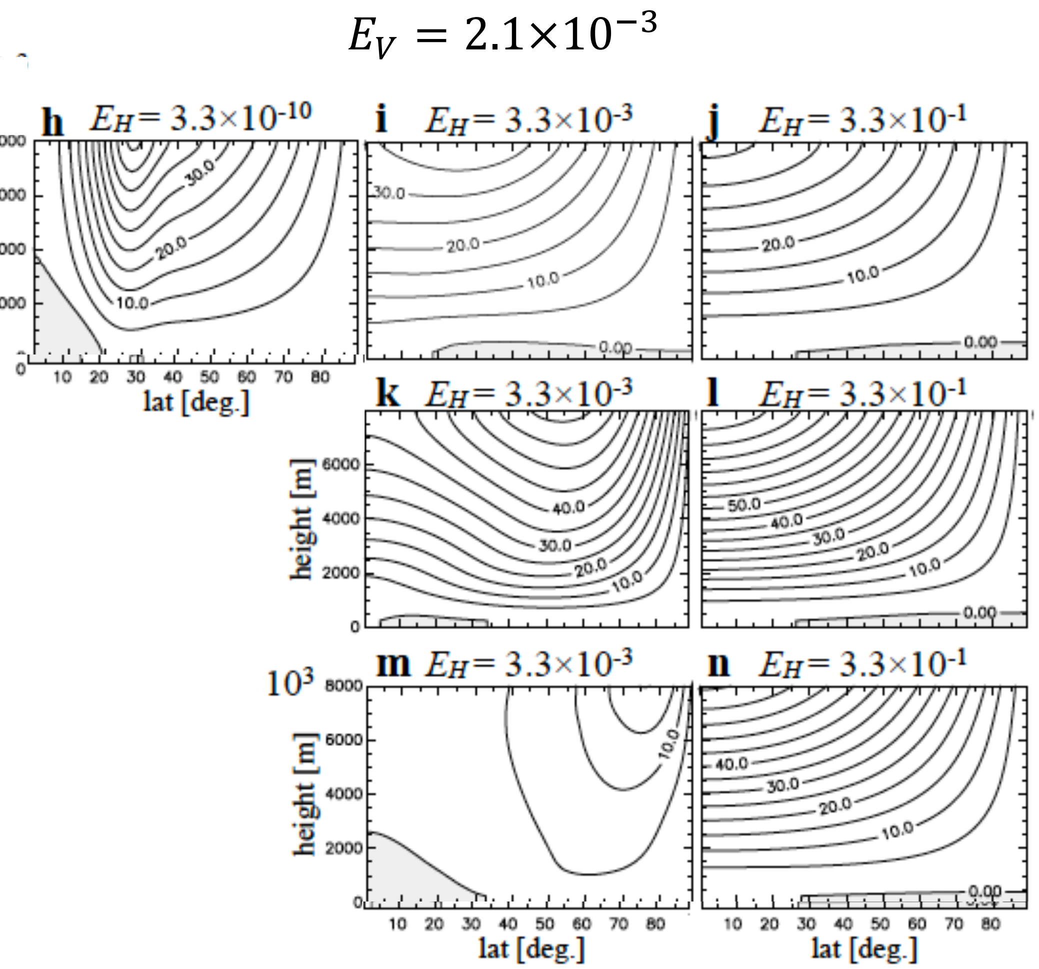
$\mathcal{R} = 0.12$

$\mathcal{R}$ = 12

$\mathcal{R}$ = 1200



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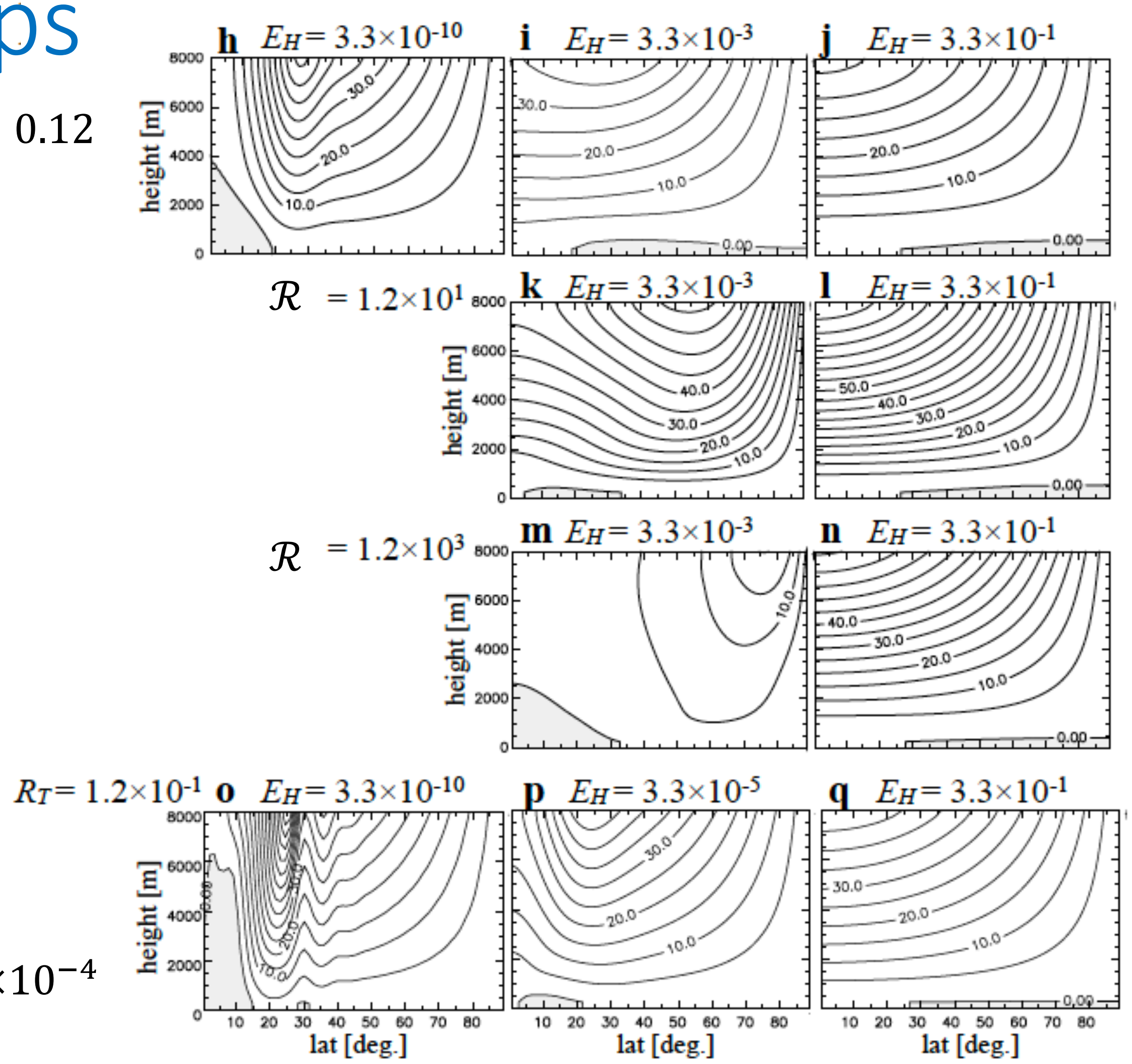

Simple theoretical/numerical models: Parameter sweeps

- Scale analysis and steady state solution of axisymmetric primitive equations with anisotropic viscosity, assuming simple spatial structures of key variables
- Reduce parameter dependence to an algebraic equation in $S' = \frac{3S}{4}$

$$\left[S'^2 + 2S' + BS'\left(\frac{2+S'}{1+S'}\right)\right]\left[\frac{AS'}{2}\left(\frac{2+S'}{1+S'}\right) + 1\right] = 2\mathcal{R}$$

- Where $A = \pi^2\tau_{rad}/\tau_V$ and $B = 20\pi^2\left(\frac{1}{\Omega\sqrt{\tau_V\tau_H}}\right)^2$ and

  τ_{rad} = radiative timescale; $(\tau_H, \tau_V) = \left(\frac{a^2}{\nu_H}, \frac{H^2}{\nu_V}\right)$

[Yamamoto & Yoden 2013; Lewis et al. 2021]

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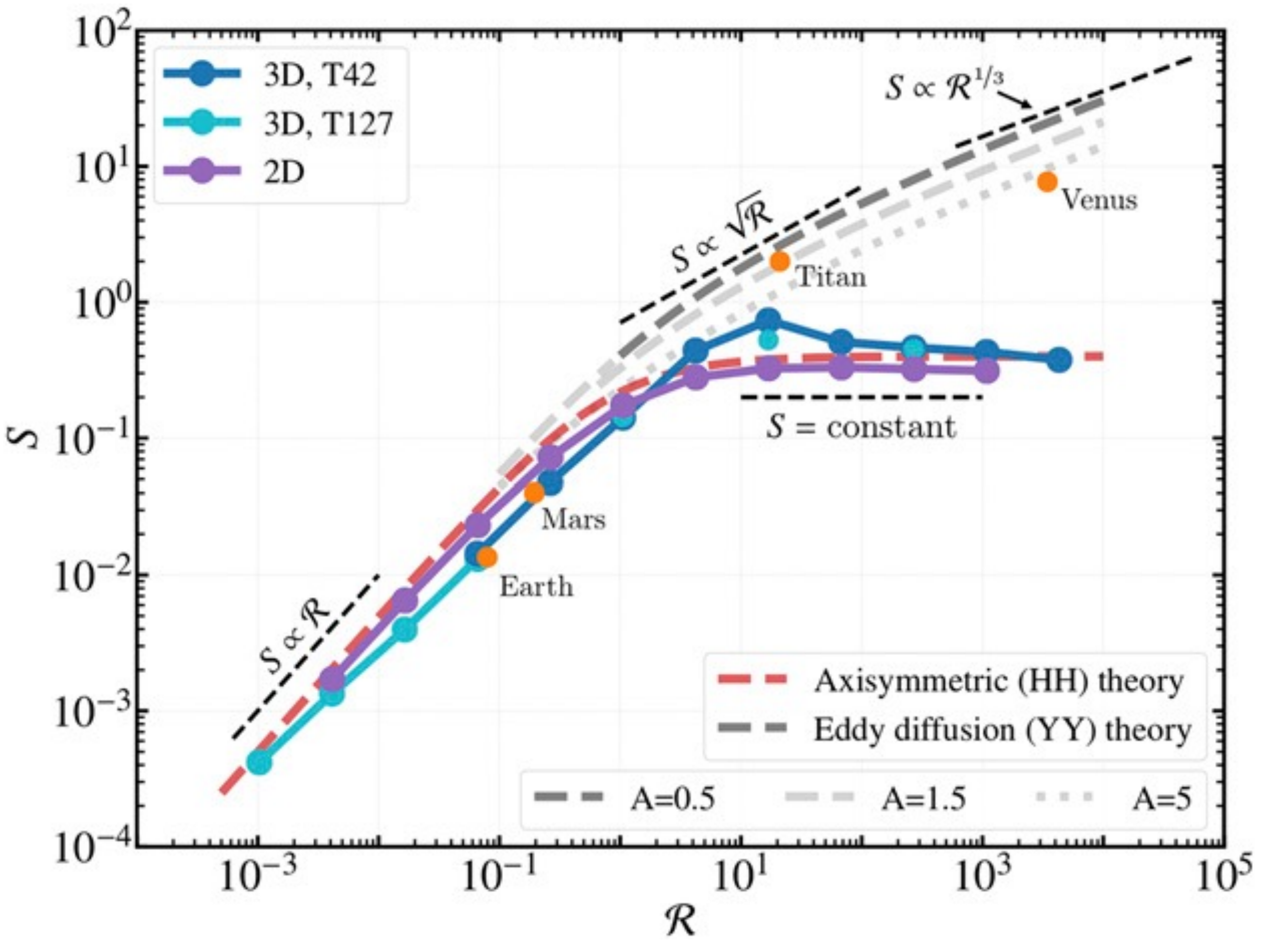
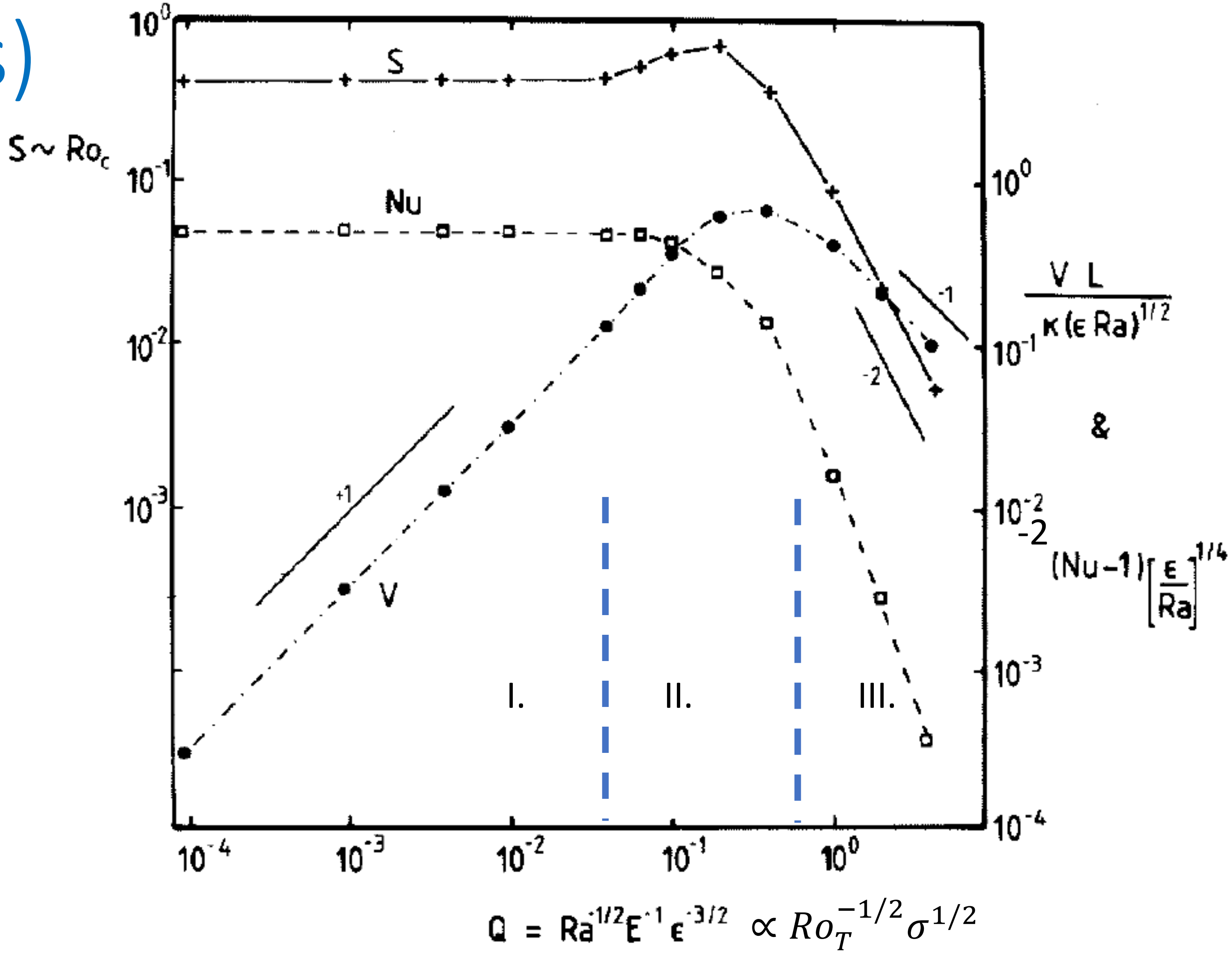
- Asymptotic dependences
  - $\mathcal{R} \ll 1$: $S \propto \mathcal{R}$ - Geostrophic flow
  - $\mathcal{R} > 1$: $S \propto \mathcal{R}^{1/2}$ - Cyclostrophic flow
  - $\mathcal{R} \geq 10$: $S \propto \mathcal{R}^{1/3}$ - Modified cyclostrophic flow



[Yamamoto & Yoden 2013; Lewis et al. 2021]

Trends in S (cylindrical annulus)

- 3 basic regimes
  - I. V. Slow rotation (~angular momentum conserving except in Ekman layer): S ~ constant
  - II. Moderate rotation (cyclostrophic/gradient wind and diffusive interior): S rises to shallow peak ($S \approx \varepsilon \eta^{1/2} \sigma^{-1/2} Q^{-1}$)
  - III. Rapid rotation (quasi-geostrophic): S ~ Q^{-2} ~ Ω^{-2}



How do eddies ACTUALLY mix & transport angular momentum?

- Which quantity to homogenize?
  - Angular momentum?
  - Vorticity?
    - [Relative, absolute or potential?]
  - Angular velocity/tangential stress?
  - Buoyancy?
  - …..
- Depends strongly on the mechanism generating the eddies…..
  - Instability?
  - Direct forcing…?

How do eddies ACTUALLY mix & transport angular momentum?

- Angular Momentum mixing, for example, leads to (relative) vorticity expulsion
    - [Gough & Lynden-Bell 1968]
- This mixes towards a state in which

$$m = (\Omega r + u).r = const.$$

Or

$$\boldsymbol{\Omega}\times\boldsymbol{r} + \boldsymbol{u} = \frac{const.}{r}\boldsymbol{e}_\theta$$

- Hence $\nabla\times\boldsymbol{u} = -2\Omega\boldsymbol{k}$ and the end-state is irrotational
- Demonstrated (roughly!) in a simple experiment?
    - Inner beaker observed to increase its rotation

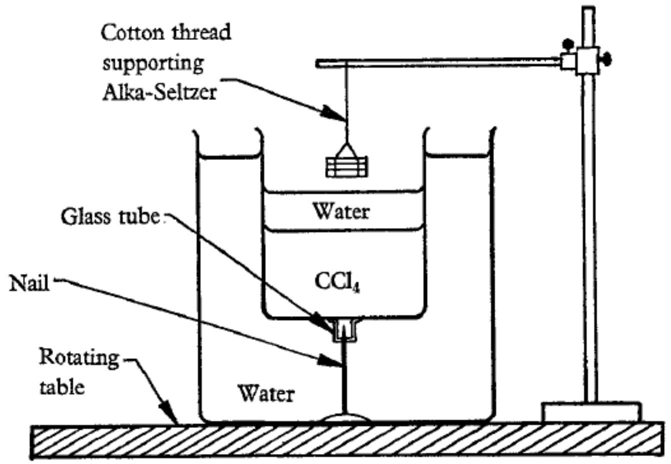


FIGURE 1. Diagram of the apparatus in its original form.

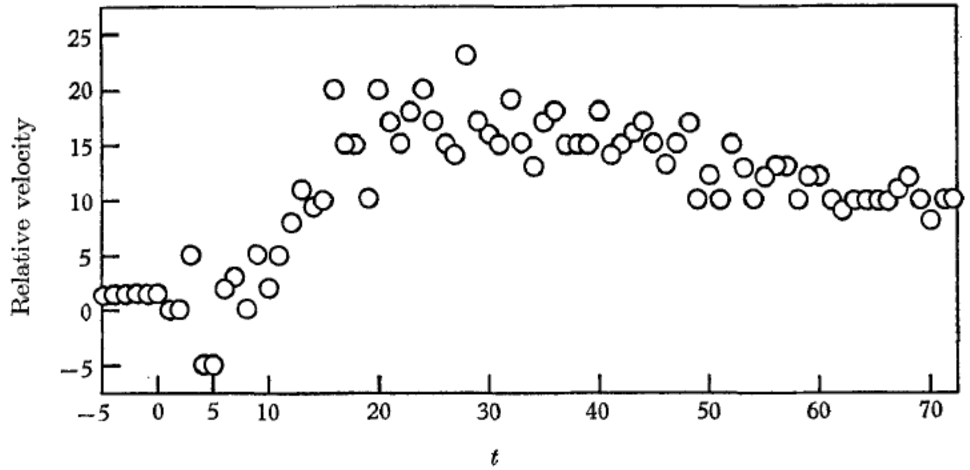
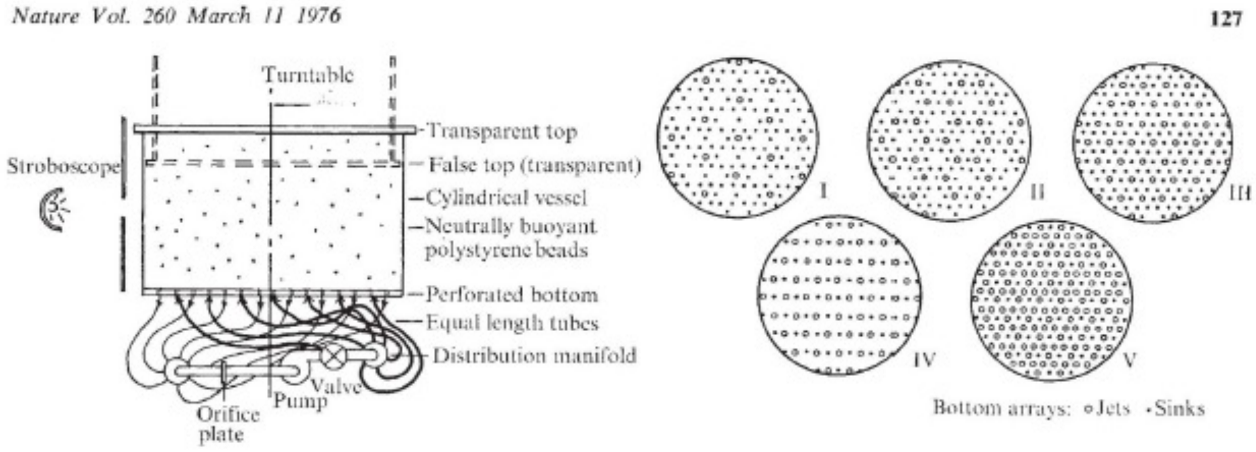


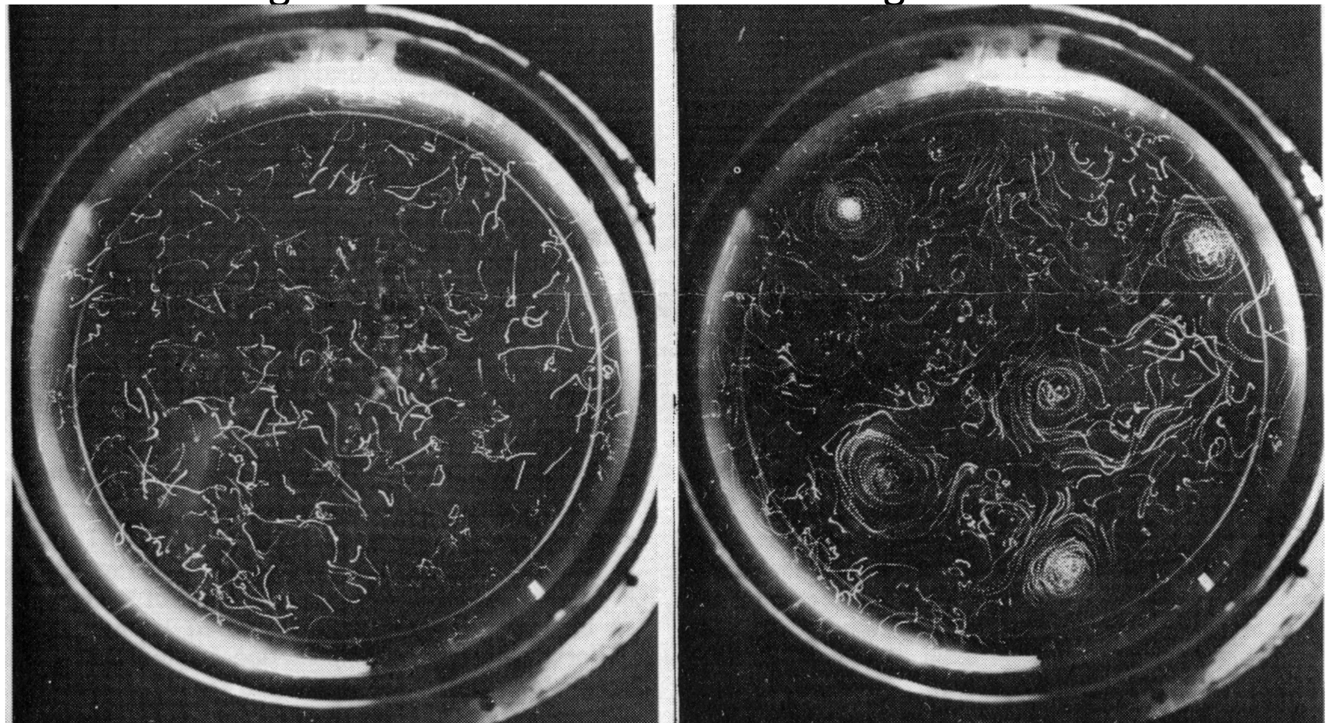
FIGURE 3. Variation of the relative velocity of the inner beaker with time. Time is measured in periods of the inner beaker and velocity in degrees per period. The Alka-Seltzer was dropped at $t = 0$.

How do eddies ACTUALLY mix & transport angular momentum?

- More generally vorticity expulsion in randomly stirred flow acts locally but not globally
  - With background rotation, initial solid body rotation with stirring forms intense cyclonic vortices
  - [e.g. McEwan 1976 Nature]
  - -> Local concentrations of vorticity
  - Without background rotation, however, no tendency to form vortices…..



No background rotation With background rotation



How do eddies ACTUALLY mix & transport angular momentum?

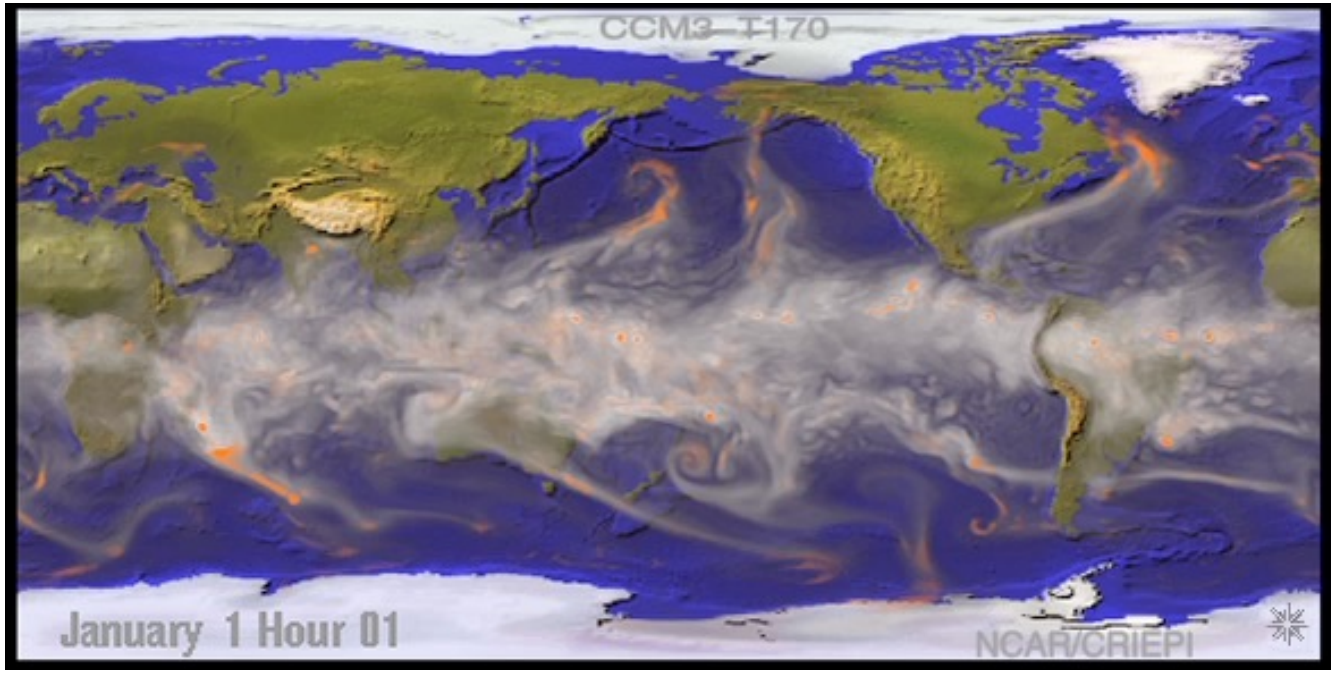
- Vorticity mixing is another possibility, especially for rotating barotropic flows
  - Taylor (1915); Rossby (1947); Eady (1950)
- For an axisymmetric (or zonally averaged) flow, absolute vorticity ξ is related to *m* by

$$\xi = \frac{1}{r}\frac{\partial m}{\partial r} = 2\Omega + \omega + \frac{\partial u}{\partial r}$$

- Mixing to a state of uniform ξ consistent with uniform angular velocity (though not uniquely so)
  - Cf viscous diffusion releasing tangential strain…..?
  - See also Rossow & Williams (1979)….vorticity mixing associated with barotropic instability

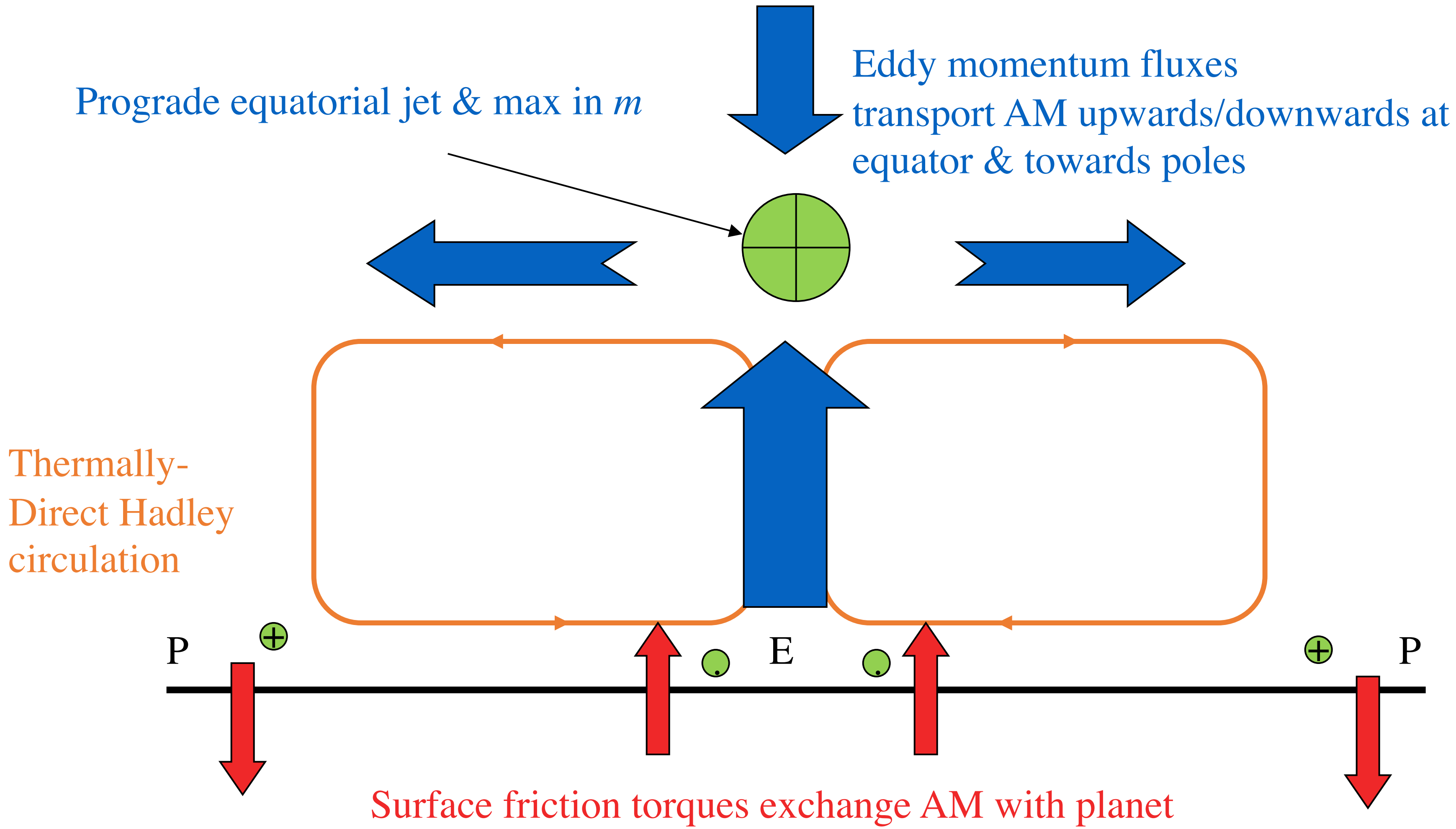
How do eddies ACTUALLY mix & transport angular momentum?

- Nonaxisymmetric eddies in a stratified atmosphere on a rotating planet are not uniformly random motions!
- Dominated by waves, at least at large scales
- But what kind of waves…?



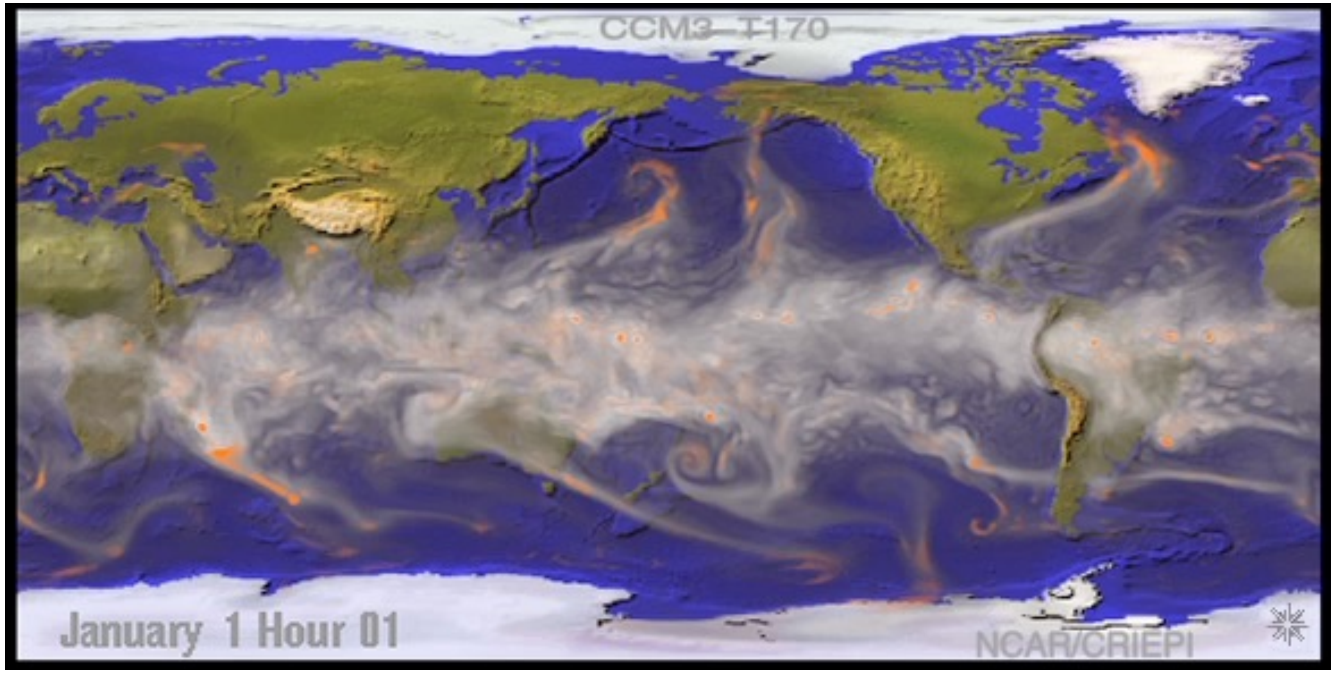
Super-rotation: an alternative non-classical GRW scenario

- An alternative scenario in which up-gradient AM transfer takes place in the ***vertical***
- Still satisfies the Starr-Hide theorem provided horizontal AM transfers are ***down-gradient***
- Not like a viscosity – needs the action of vertically propagating waves….?



How do eddies ACTUALLY mix & transport angular momentum?

- Nonaxisymmetric eddies in a stratified atmosphere on a rotating planet are not uniformly random motions!
- Dominated by waves, at least at large scales
- But what kind of waves…?



Start with linearised primitive equations for a stratified atmosphere at rest on a spherical planet

- $\frac{\partial u'}{\partial t} - fv' + \frac{1}{a\cos\varphi}\frac{\partial \Phi'}{\partial \lambda} = X'$ zonal momentum
- $\frac{\partial v'}{\partial t} + fu' + \frac{1}{a}\frac{\partial \Phi'}{\partial \varphi} = Y'$ meridional momentum
- $\frac{1}{a\cos\varphi}\left[\frac{\partial u'}{\partial \lambda} + \frac{\partial}{\partial \varphi}(v'\cos\varphi)\right] + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w') = 0$ continuity
- $\frac{\partial^2 \Phi'}{\partial z \partial t} + N^2 w' = \frac{\kappa J'}{H}$ Internal energy
- With boundary conditions such as
  - $\frac{\partial \Phi'}{\partial t} = -w'\frac{\partial \bar{\Phi}}{\partial z} = \frac{R\bar{T}(0)w'}{H}$ at $z=0$ [for a flat lower boundary]

$f = 2\Omega \sin\varphi$
Φ = geopotential
X', Y' = friction or forcing
J = diabatic heating
H = scale height

Look for separable solutions of the form

$$(u', v', \Phi') = e^{\frac{z}{2H}} U(z) \left[\tilde{u}(\lambda, \varphi, t), \tilde{v}(\lambda, \varphi, t), \tilde{\Phi}(\lambda, \varphi, t) \right] \quad (*)$$

$$w' = e^{\frac{z}{2H}} W(z) \tilde{w}(\lambda, \varphi, t)$$

- Take $X' = Y' = J' = 0$ and substitute (*) into momentum equations to obtain

$$\frac{\partial \tilde{u}}{\partial t} - f\tilde{v} + \frac{1}{a\cos\varphi} \frac{\partial \tilde{\Phi}}{\partial \lambda} = 0 \qquad \text{(a)}$$

$$\frac{\partial \tilde{v}}{\partial t} + f\tilde{u} + \frac{1}{a} \frac{\partial \tilde{\Phi}}{\partial \varphi} = 0 \qquad \text{(b)}$$

- Separate z-dependence via a separation constant $1/gh$ to obtain

$$\frac{1}{a\cos\varphi} \left[\frac{\partial \tilde{u}}{\partial \lambda} + \frac{\partial}{\partial \varphi} (\tilde{v}\cos\varphi) \right] + \frac{1}{gh} \frac{\partial \tilde{\Phi}}{\partial t} = 0 \qquad \text{(c)}$$

[Equivalent depth *h*]

Laplace Tidal Equations

$$\frac{\partial \tilde{u}}{\partial t} - f\tilde{v} + \frac{1}{a\cos\varphi}\frac{\partial \tilde{\Phi}}{\partial \lambda} = 0 \qquad \text{(a)}$$

$$\frac{\partial \tilde{v}}{\partial t} + f\tilde{u} + \frac{1}{a}\frac{\partial \tilde{\Phi}}{\partial \varphi} = 0 \qquad \text{(b)}$$

$$\frac{1}{a\cos\varphi}\left[\frac{\partial \tilde{u}}{\partial \lambda} + \frac{\partial}{\partial \varphi}(\tilde{v}\cos\varphi)\right] + \frac{1}{gh}\frac{\partial \tilde{\Phi}}{\partial t} = 0 \qquad \text{(c)}$$

- Together with vertical structure equations

$$\frac{d^2W}{dz^2} + \left(\frac{N^2}{gh} - \frac{1}{4H^2}\right)W = 0; \quad U = \frac{dW}{dz} - \frac{W}{2H} \qquad \text{(d,e)}$$

- And suitable boundary conditions

Now look for separable solutions of the form

$$(u', v', \Phi') = Re\left[\hat{u}(\varphi), \hat{v}(\varphi), \widehat{\Phi}(\varphi)\right]\exp i(s\lambda - 2\Omega\sigma t)$$

- Where s is zonal wavenumber index and $2\pi/2\Omega\sigma$ is the wave period (or $\pi/2\sigma$ is the period in days).
- Substituting the solution into (a)-(c) and eliminating $\hat{u}(\varphi)$ and $\hat{v}(\varphi)$ leads to "Laplace's Tidal Equation"

$$\mathcal{L}\widehat{\Phi} + \gamma\widehat{\Phi} = 0 \qquad \text{(f)}$$

- Where $\gamma = \frac{4\Omega^2 a^2}{gh}$ is Lamb's parameter and $\mathcal{L}$ is a second order differential operator in $\mu = \sin\varphi$

$$\mathcal{L} = \frac{d}{d\mu}\left[\frac{(1-\mu^2)}{(\sigma^2-\mu^2)}\frac{d}{d\mu}\right] - \frac{1}{\sigma^2-\mu^2}\left[\frac{s^2}{1-\mu^2} - \frac{s(\sigma^2+\mu^2)}{\sigma(\sigma^2-\mu^2)}\right]$$

- With boundary conditions that $\widehat{\Phi}$ is bounded at the poles ($\mu = \pm 1$)

Solve (f) as an eigenvalue problem

- Prescribe s and σ and solve (f)

$$\mathcal{L}\Theta_n^{(\sigma,s)} + \gamma_n^{(\sigma,s)}\Theta_n^{(\sigma,s)} = 0$$

for different integer values of n

- Θ_n are Hough functions
  - E.g. see Longuet-Higgins (1968)
- Dispersion relations computed as function of $\gamma^{-1/2}$

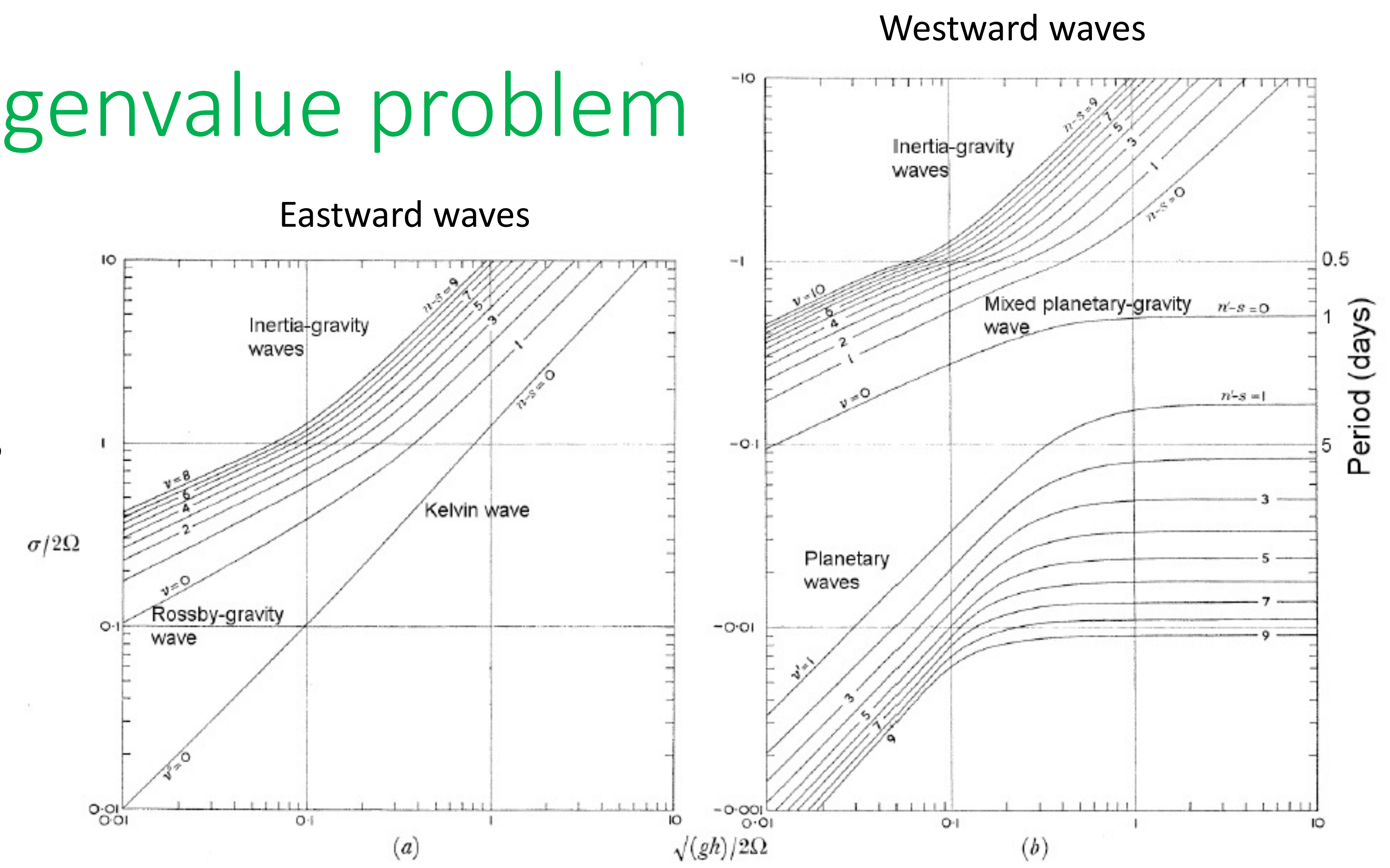
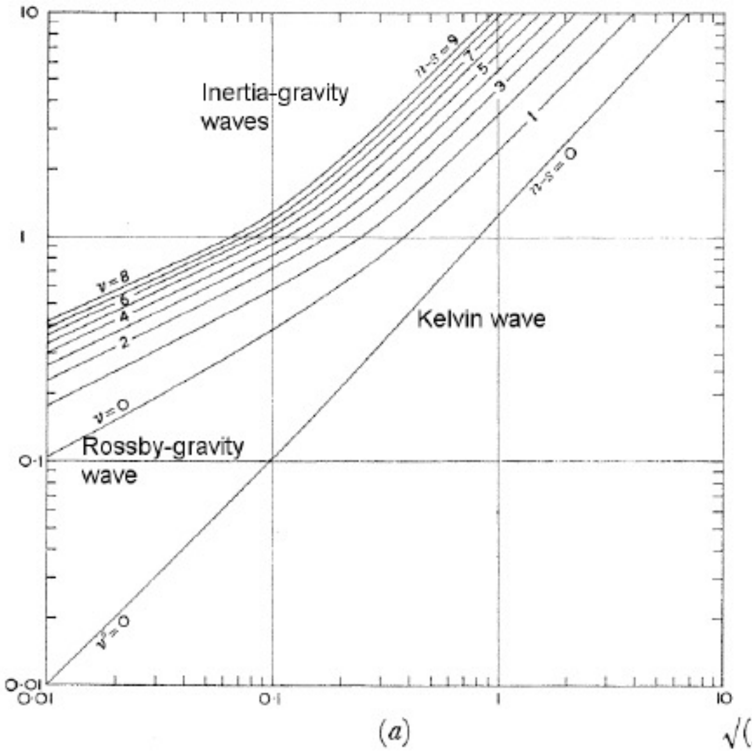


Figure 5.12: Plots of $(g|h|)^{1/2}/(2\Omega a)$ versus frequency σ for zonal wavenumber $s = 1$ in the linearised Laplace tidal equations for positive equivalent depth h: (a) eastward phase speeds ($\sigma > 0$) and (b) westward phase speeds ($\sigma < 0$). Figure adapted from *Longuet-Higgins* [1968].

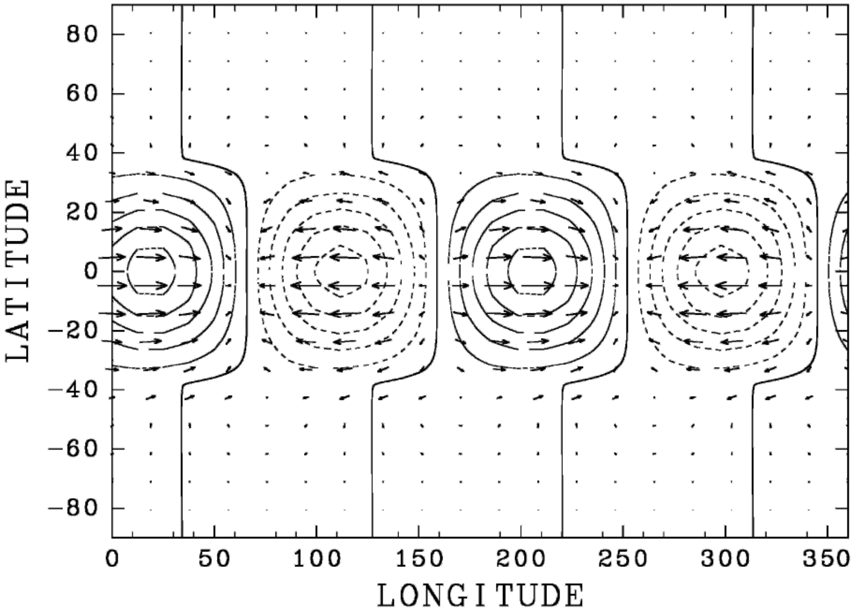
Free atmospheric waves

- Several families of waves
  - Eastward modes include high frequency inertia-gravity modes and the equatorial Kelvin mode
  - Westward modes also include high frequency inertia-gravity waves and low frequency planetary Rossby waves (ν'≥1) and the mixed planetary-gravity mode (ν=0)
- Asymptotic limits
  - Low rotation $\left(\gamma^{-1/2} \rightarrow \infty\right)$ waves are global in extent
  - Fast rotation $\left(\gamma^{-1/2} \rightarrow 0\right)$ waves are trapped either near the equator (*h* > *0*) or the poles (*h* < *0*)

Eastward waves



Kelvin wave



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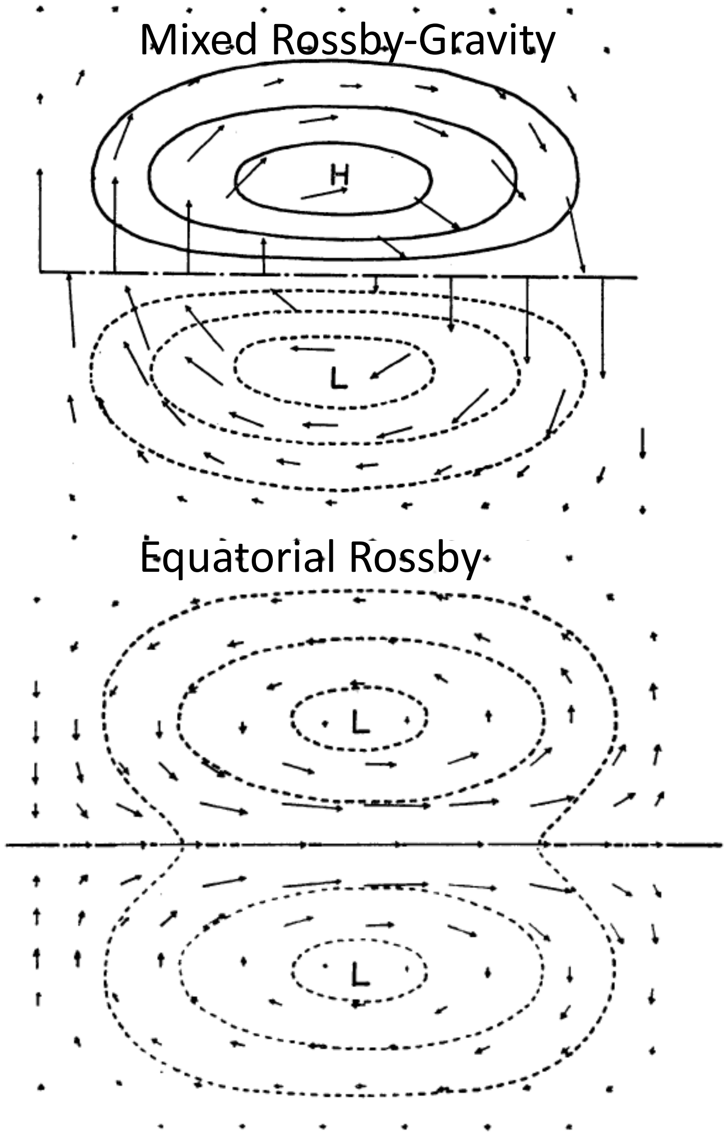
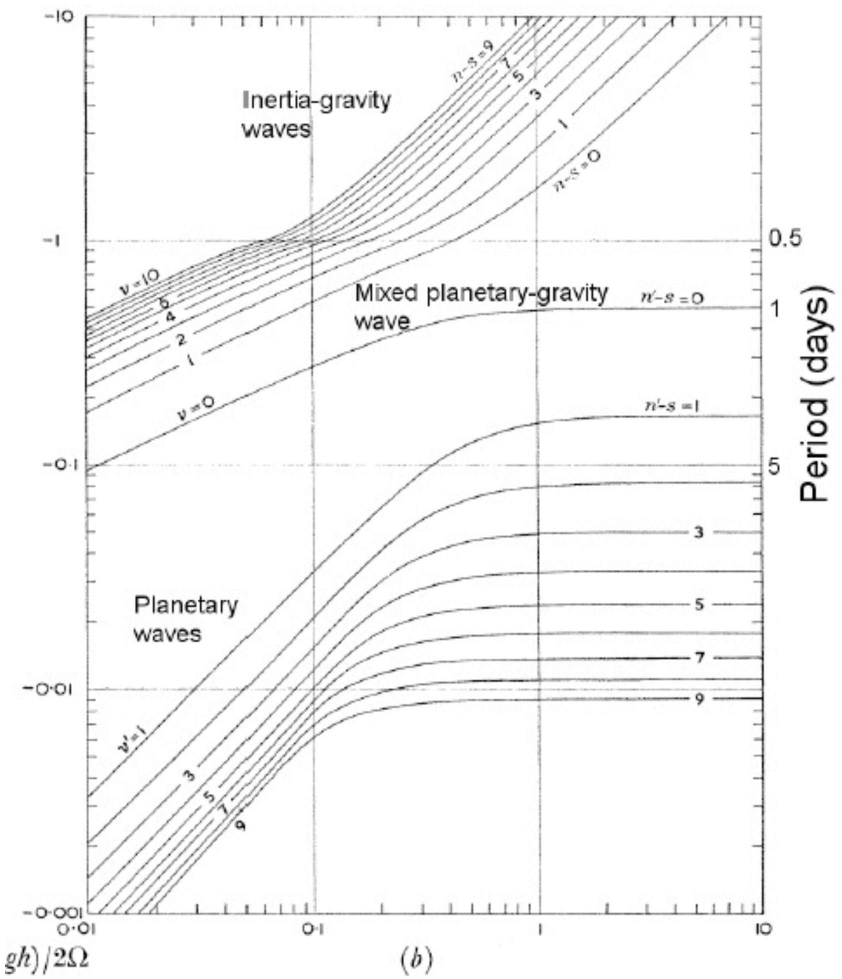


Westward waves



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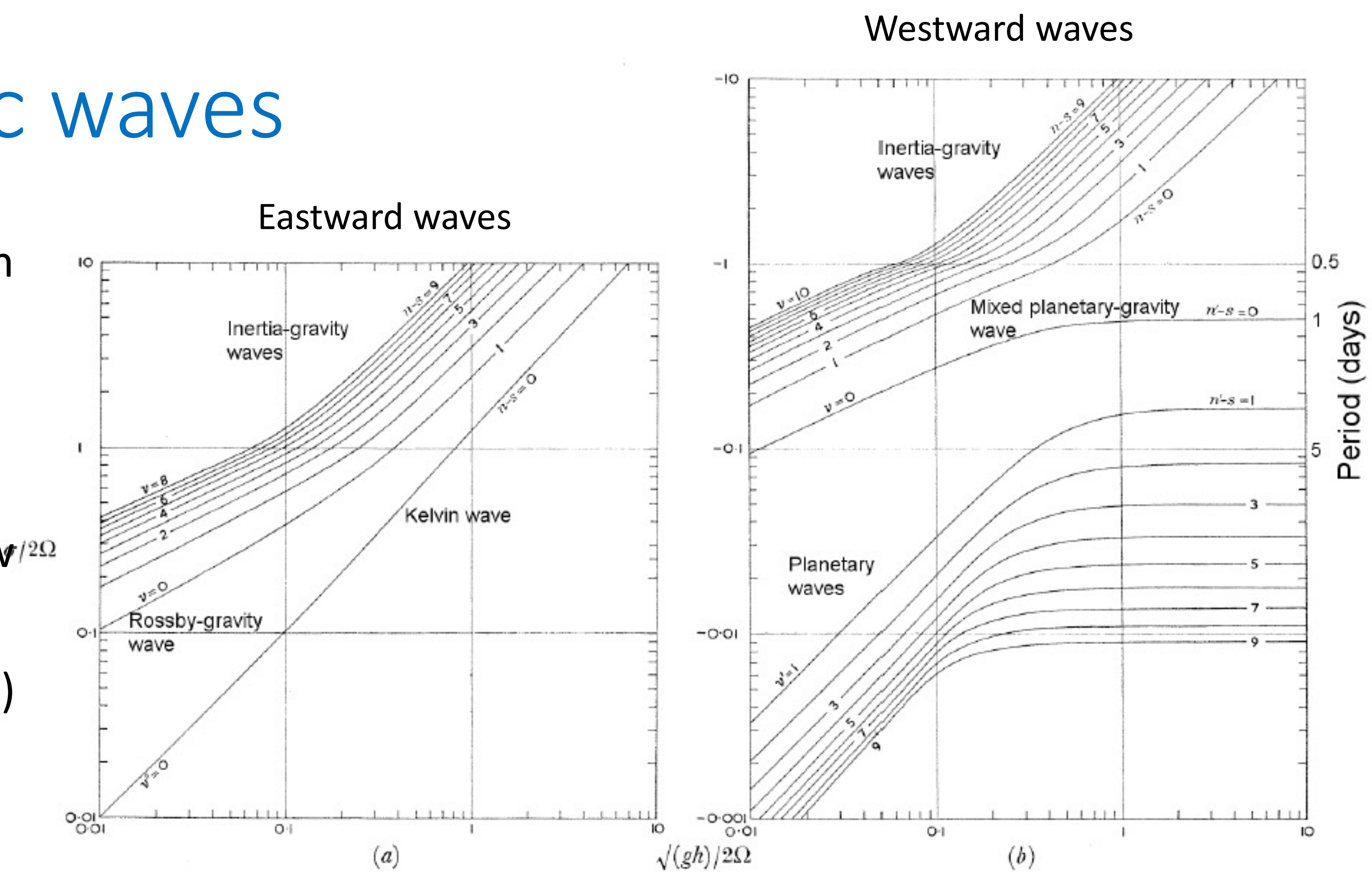

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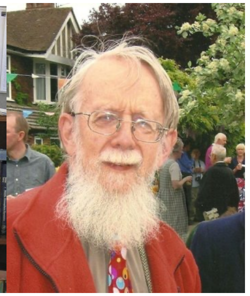
Summary & remarks

- For a stationary atmosphere (co-rotating with a spherical planet), there exists a well defined spectrum of freely-evolving (neutrally stable) waves
  - Various combinations of stratification/buoyancy and inertial/Coriolis restoring forces
  - Propagate dispersively in the zonal direction
  - Characteristic and distinctive spatial structures
  - Propagate dispersively (and anisotropically) also in the vertical with wavenumber $m_n^{(s,\sigma)}$
  - Phase and group velocities, and spatial structures, depend on Lamb parameter $\gamma^{-1/2} = \sqrt{g|h|}/(\Omega a)$ [cf thermal Rossby $\mathcal{R}$ and Burger $\mathcal{B}$ numbers]

Used to classify each wave mode

- Asymptotic forms are well approximated for fast rotation $\left(\gamma^{-1/2} \rightarrow 0\right)$ by tangent-plane models
  - E.g. equatorial or mid-latitude β-planes
- Wave structures and dispersion relations are significantly modified in the presence of a zonally averaged shear flow $[\bar{U}(\varphi, z, t) \neq 0]$, dissipation, forcing and nonlinearity
  - Nevertheless, the classification and nomenclature of the linear, conservative waves are still used to classify similar wave types in other regimes.....

How and when do waves accelerate or decelerate zonal flows?



- Wave-zonal mean interaction theory [Andrews & McIntyre 1976, 1978]
- Based on zonal mean primitive equations
- $\frac{\partial \bar{u}}{\partial t} + \bar{v}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\bar{u}\cos\varphi) - f\right] + \bar{w}\frac{\partial \bar{u}}{\partial z} - \bar{X} = -\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\overline{u'v'}cos^2\varphi\right) - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\overline{w'u'}\right)$
- $\frac{\partial \bar{v}}{\partial t} + \frac{\bar{v}}{a}\frac{\partial \bar{v}}{\partial\varphi} + \bar{w}\frac{\partial \bar{v}}{\partial z} + \bar{u}\left(f + \frac{\bar{u}\tan\varphi}{a}\right) + \frac{1}{a}\frac{\partial\bar{\Phi}}{\partial\varphi} - \bar{Y} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\overline{v'^2}\right) - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\overline{w'v'}\right) - \frac{\overline{u'^2}\tan\varphi}{a}$
- $\frac{\partial\bar{\Phi}}{\partial z} - \frac{R}{H}\bar{\theta}e^{-\frac{\kappa z}{H}} = 0$
- $\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\bar{v}\cos\varphi) + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar{w}) = 0$
- $\frac{\partial\bar{\theta}}{\partial t} + \frac{\bar{v}}{a}\frac{\partial\bar{\theta}}{\partial\varphi} + \bar{w}\frac{\partial\theta}{\partial z} - \bar{Q} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\overline{v'\theta'}\cos\varphi\right) - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\overline{w'\theta'}\right)$

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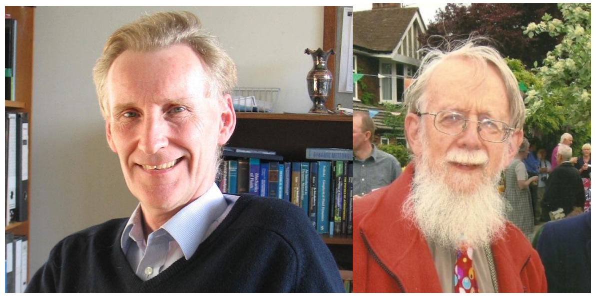

- Problem: how to separate zonal mean circulation from eddies?
  - Eddies also influence Eulerian zonal mean meridional flow $(\bar{v}, \bar{w})$
  - Eddy heat fluxes can also implicitly influence zonal momentum
- Instead, use Transformed Eulerian Mean (TEM) equations

- $\frac{\partial \bar{u}}{\partial t} + \overline{v^*}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\bar{u}\cos\varphi) - f\right] + \overline{w^*}\frac{\partial \bar{u}}{\partial z} - \bar{X} = -\frac{1}{a\cos\varphi}\boldsymbol{\nabla}.\boldsymbol{F}$
- $\bar{u}\left(f + \frac{\bar{u}\tan\varphi}{a}\right) + \frac{1}{a}\frac{\partial\bar{\Phi}}{\partial\varphi} = G$
- $\frac{\partial\bar{\Phi}}{\partial z} - \frac{R}{H}\bar{\theta}e^{-\frac{\kappa z}{H}} = 0$
- $\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\overline{v^*}\cos\varphi) + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\overline{w^*}) = 0$
- $\frac{\partial\bar{\theta}}{\partial t} + \frac{\overline{v^*}}{a}\frac{\partial\bar{\theta}}{\partial\varphi} + \overline{w^*}\frac{\partial\theta}{\partial z} - \bar{Q} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left[\rho_0\left(\frac{\overline{v'\theta'}\partial\bar{\theta}/\partial\varphi}{a\partial\bar{\theta}/\partial z} + \overline{w'\theta'}\right)\right]$

Where Residual Mean Circulation is

- $\overline{v^*} = \bar{v} - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0\overline{v'\theta'}}{\partial\bar{\theta}/\partial z}\right)$
- $\overline{w^*} = \bar{w} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\overline{v'\theta'}\cos\varphi}{\partial\bar{\theta}/\partial z}\right)$

And Eliassen-Palm Flux ***F*** is

- $F^{(\varphi)} = \rho_0 a\cos\varphi\left(\frac{\partial\bar{u}}{\partial z}\frac{\overline{v'\theta'}}{\partial\bar{\theta}/\partial z} - \overline{u'v'}\right)$
- $F^{(z)} = \rho_0 a\cos\varphi\left\{\left[f - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\bar{u}\cos\varphi)\right]\frac{\overline{v'\theta'}}{\partial\bar{\theta}/\partial z} - \overline{u'w'}\right\}$

Remarks on TEM

- Separates eddies and zonal mean circulations more "cleanly"
  - Residual mean circulation approximates to a Lagrangian Mean circulation under some circumstances
- The Eliassen-Palm (EP) flux ***–F*** represents an effective eddy flux of angular momentum
  - So $\boldsymbol{\nabla}.\boldsymbol{F}$ represents a ***convergence*** of AM flux, producing an eastward tendency for $\bar{u}$ from eddy stresses
  - In quasi-geostrophic flows, ***F*** is simplified [terms in blue] and $\boldsymbol{\nabla}.\boldsymbol{F} = \overline{v'q'}$
- EP fluxes are directly associated with the so-called Eliassen-Palm theorem (Eliassen & Palm 1961) and generalisations by Boyd (1976) and Andrews & McIntyre (1976,1978), and the Charney-Drazin non-acceleration theorem (Charney & Drazin 1961).

Generalised EP Theorem

- Eliassen & Palm (1961) considered steady, linear waves on a basic zonal flow $\bar{u}(\varphi, z)$ with no frictional or diabatic effects (on a β-plane)
  - They showed that under these conditions $\boldsymbol{\nabla}.\boldsymbol{F} = 0$
- This result was extended to non-zero *X′*, *Y′* and *Q′*, and to spherical geometry, by Boyd (1976) and to time-varying waves by Andrews & McIntyre (1976,1978).
- A&M showed that

$$\frac{\partial A}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{F} = D + O(\alpha^3)$$

  - *A* is "wave activity density" and *D* represents frictional and diabatic effects.
  - [$A \approx \frac{1}{2}\rho_0 \overline{q'^2}/(\partial\bar{q}/\partial y)$ for quasi-geostrophic problems]
  - α represents wave amplitude so this term represents higher order nonlinearity
  - Makes explicit the dependence of EP flux divergence on wave transience and non-conservative effects

Generalised EP Theorem

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$$\frac{\partial A}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{F} = D + O(\alpha^3) \qquad (*)$$

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  - α represents wave amplitude so this term represents higher order nonlinearity
  - Makes explicit the dependence of EP flux divergence on wave transience and non-conservative effects

- Note that (*) can also be written in the absence of friction and diabatic effects under some conditions as

$$\frac{\partial A}{\partial t} = -\boldsymbol{\nabla}.(\boldsymbol{c_g}A) + O(\alpha^3)$$

where $\boldsymbol{c_g}$ is the group velocity, so $\boldsymbol{F}$ is related to group velocity

Non-acceleration Theorem

- TEM equations indicate that eddies will accelerate/decelerate zonal flow if $\boldsymbol{\nabla}.\boldsymbol{F} \neq 0$
- But the Generalised EP theorem shows that $\boldsymbol{\nabla}.\boldsymbol{F} = 0$ for steady, linear and conservative waves
  - Hence, under those conditions a possible mean flow satisfying the full TEM equations could be
  - $\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{\theta}}{\partial t} = \overline{v^*} = \overline{w^*} = 0.$ i.e. a non-accelerating solution
  - [Given appropriate boundary conditions]
- The corollary of this is that wave transience, dissipation or forcing may lead to $\boldsymbol{\nabla}.\boldsymbol{F} \neq 0$ and an acceleration of a mean zonal flow

Non-acceleration Theorem: an illustration

- TEM equations indicate that eddies will accelerate/decelerate zonal flow if $\boldsymbol{\nabla}.\boldsymbol{F} \neq 0$
- Illustrate with an analytic example consisting of a linear superposition of a QG finite-amplitude, baroclinic/barotropic wave and zonal flow with diabatic heating
  - N^2 assumed uniform
  - Formulated to be non-interacting free modes so $\boldsymbol{v}_g.\nabla Q = 0$
  - i.e. $\overline{v'Q'} = 0 = \nabla.\boldsymbol{F}$ *even though individual momentum and heat fluxes are non-zero*

[Read 1985]

Ψ

Z = 0.75H

Z=0.5H

Z=0.25H

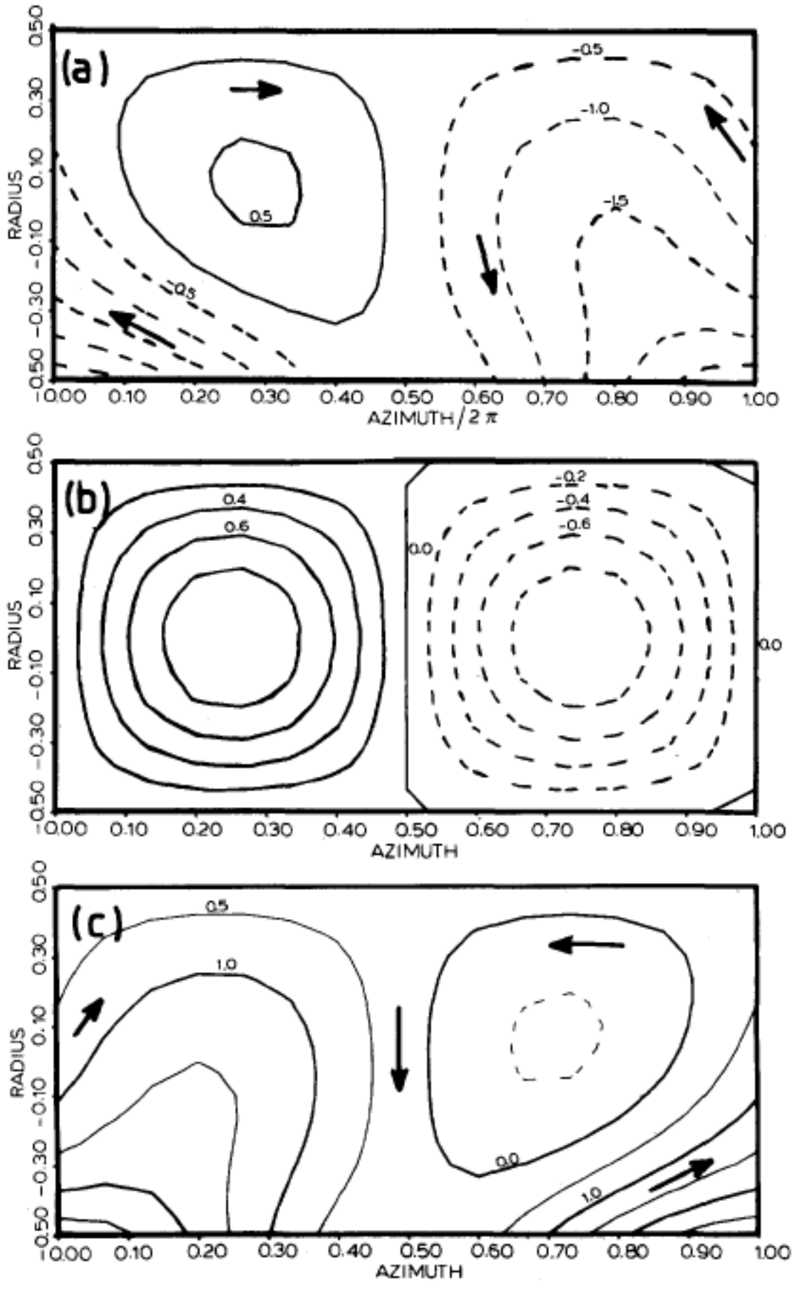

Fig. 24.

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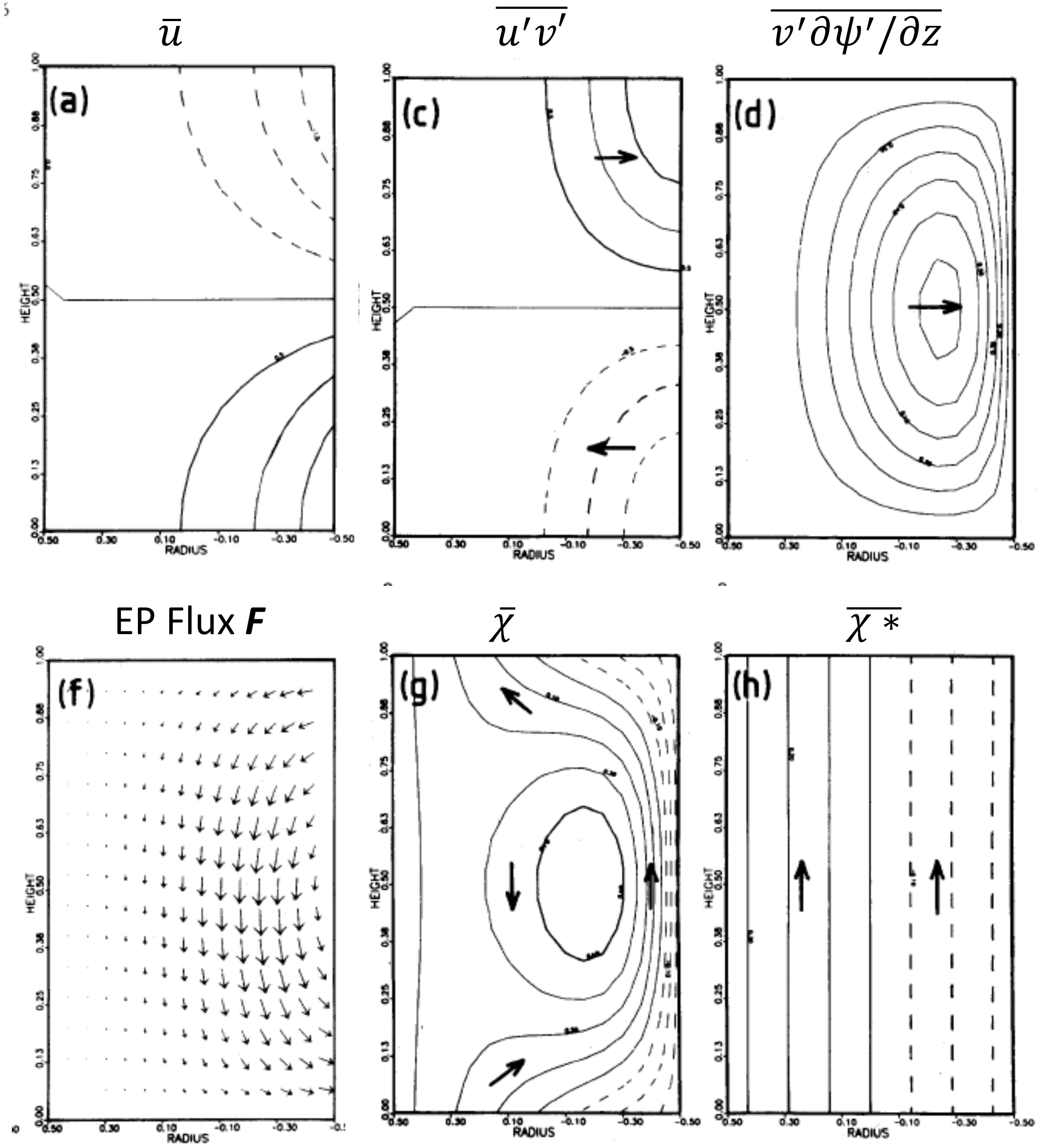

Critical layer absorption (a quick summary!)

- Propagation of Rossby or inertia-gravity waves may be strongly affected in regions where phase speed $c \approx \bar{u}$
  - a CRITICAL LAYER or LINE
- Group velocity c_g may become small perpendicular to critical line
- Reduced wavelength normal to critical line -> increased dissipation and nonlinearity
  - E.g. breaking and overturning of wave
- Absorption and dissipation of wave pseudo-momentum may lead to enhanced acceleration of $\bar{u}$ near critical layer
- Hence $\bar{u}$ tends to be accelerated towards phase speed c as wave dissipates

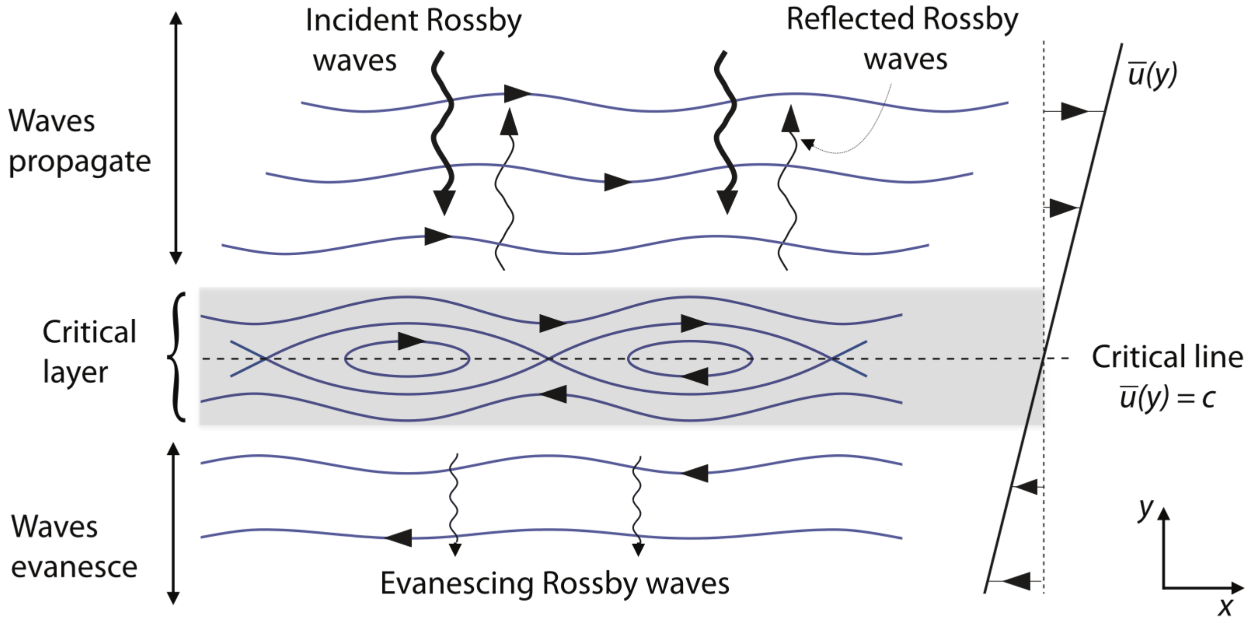

Fig. 16.5

Credit: Vallis [2017]

Summary

- Held & Hou model captures many aspects of meridional overturning circulation
    - With some global super-rotation but without local super-rotation
- Gierasch model provides overall concept for super-rotation via diffusive eddies
- Extended to strongly viscous flows by Yamamoto & Yoden
    - can produce S ~ 10 as on Venus…..
- Leads to scaling dependence of Global super-rotation on Ω and $\mathcal{R}$
- Vorticity and its relatives is more likely to be diffused by (QG?) eddies or waves, though upward AM transport possible
- Wave-zonal flow interaction requires $\boldsymbol{\nabla}.\boldsymbol{F} \neq 0$ – wave dissipation, forcing or transience?
- Critical layers facilitate dissipation of waves and acceleration of zonal flows towards phase speed *c*