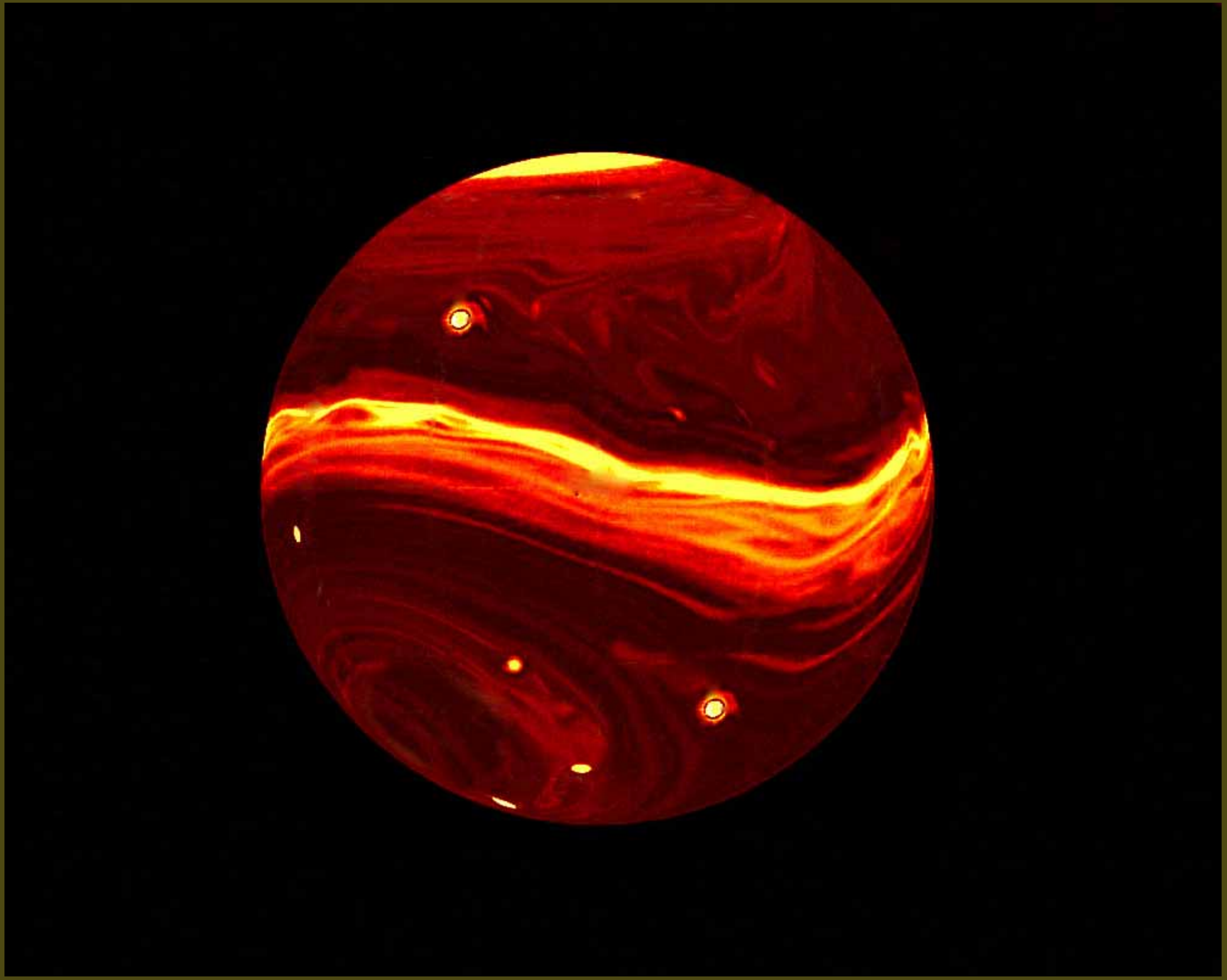


Dynamical Interactions Among Extrasolar Planets

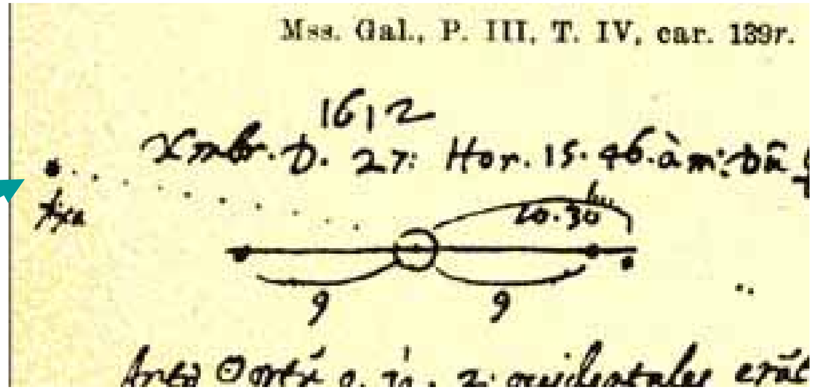
Kobe Planetary School, HJD 2453568.54

Greg Laughlin, UC Santa Cruz

www.ucolick.org/~laugh/kobe/kobe.html

Our situation today has an interesting parallel to the time when Galileo first turned his telescope to the skies in the early 1600s. Detail can be seen in the immediate neighborhood (in his case on the surface of the moon, in our case on all the planets of the solar system). The dynamics (but not the visual detail) can be sensed for distant systems (in his case the Jovian system, in our case, the extrasolar planets).

A *photograph* of a *crescent* Neptune



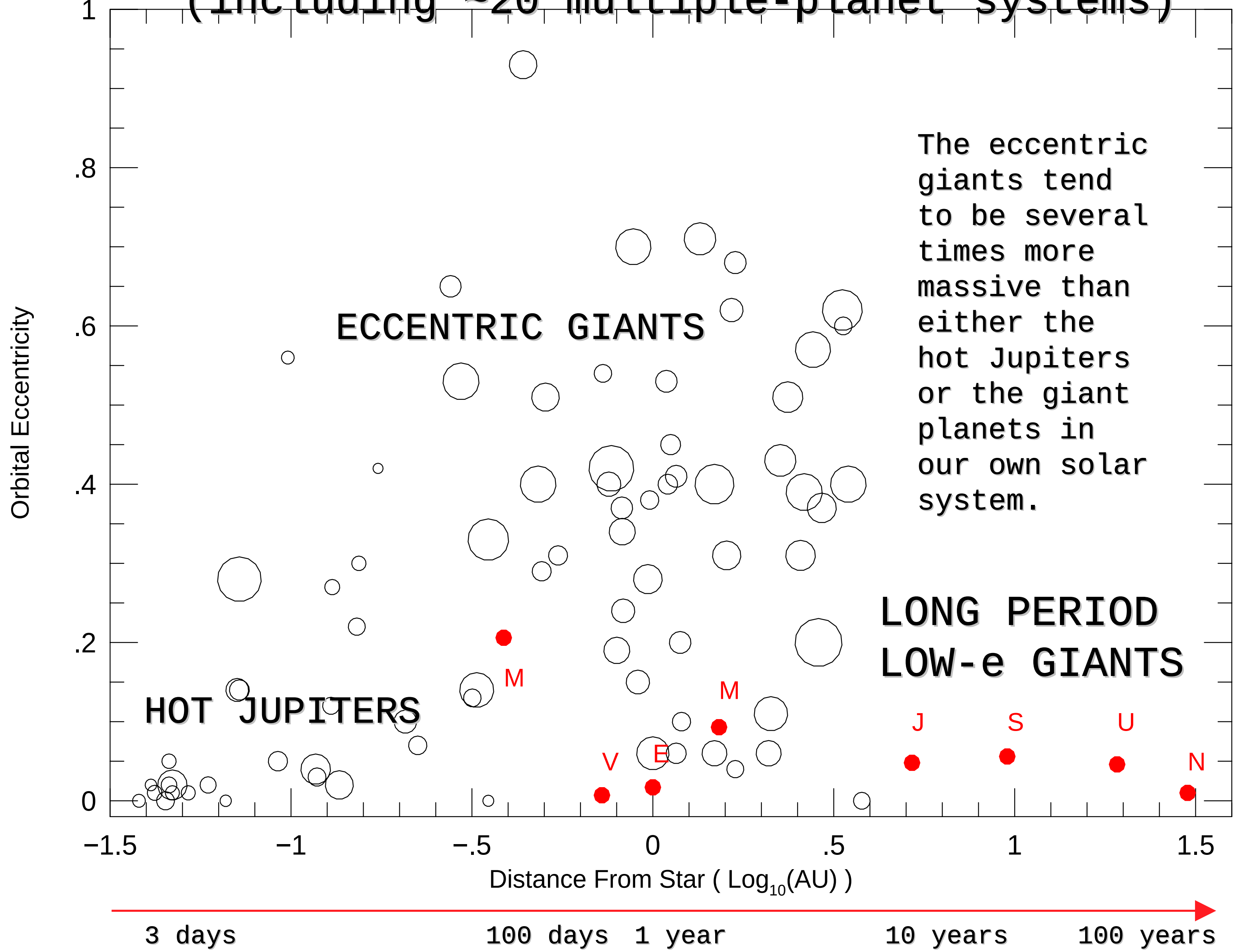
The first glimpse of Neptune
(from Galileo’s notebook)

Ben Oppenheimer talked yesterday about getting high-definition images of habitable extrasolar planets within our lifetimes. At first, that sounds rather far fetched, But I think it is actually a realistic, honest prediction.

High-definition images of extrasolar planets represent less of an advance than has been made since Galileo’s time

More astronomers are at work now, than all the astronomers of the past.

Roughly 170 Planets are now known (including ~20 multiple-planet systems)



QuickTime™ and a
Sorenson Video 3 decompressor
are needed to see this picture.

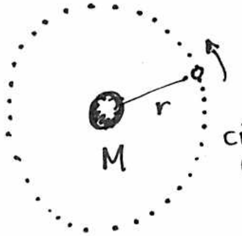
Animation of the dynamics of the precursor to the Upsilon Andromedae Planetary system (Ford et al. 2004 simulation, NSF Animation)

To consider simultaneously all these causes of motion, and to define these motions by exact laws admitting of easy calculation exceeds, if I am not mistaken, the force of any human mind.

- Isaac Newton

THE TWO BODY PROBLEM

For the problem of two bodies, we have an astronomical case of one size fits all: at the AY2 level, we have:



circular orbit → $F = ma \Rightarrow \frac{v^2}{r} = \frac{GM}{r^2}$; $2\pi r = vP \rightarrow P^2 = \frac{4\pi^2}{GM} r^3$

KEY IDEA → An orbit is a state of continuous free fall.

The apocrophal(?) story of Newton and the apple:

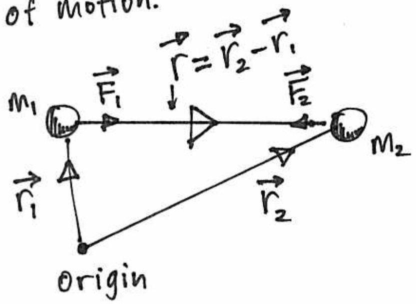
For the moon: $P^2 = \frac{4\pi^2}{GM} r^3_{\text{moon orbit}}$ (P known; $r_{\text{moon orbit}}$ known; GM determined → agreement in 1666 to 15% "pretty nearly")

For the apple: $g = \frac{GM}{r_\oplus^2}$ (g known; $r_\oplus$ known)

Colwell, P. 1993 "Solving Kepler's equation over 3 centuries" (Willmann-Bell: Richmond)

- scientific papers about the solution of the 2-body problem have been published in nearly every decade since 1650.

- $n=1$, $n=2$ are the only n-body systems for which an analytic solution describes the fully general possibilities of motion.

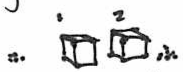


Newton II + Universal Gravitation

1. $\vec{F}_1 = +\frac{G m_1 m_2}{r^3} \vec{r} = m_1 \ddot{\vec{r}}_1$

2. $\vec{F}_2 = -\frac{G m_1 m_2}{r^3} \vec{r} = m_2 \ddot{\vec{r}}_2$

The gravitational constant $G = 6.67 \times 10^{-8} \frac{cm^3}{gm\, s^2}$.



- a sugar cube is about 1 gm and about 1 cm^3.
- The numerical value of G is telling us that two sugar cubes placed 1 cm apart in space take about $1/\sqrt{G}$ seconds $\approx$ 1 hr to come together. Natural units for gravity are thus "cgh" because they make $G \sim 1$.

$$\vec{F}_1 = -\vec{F}_2 \; ; \qquad m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = 0$$

- integrate 1 time wrt time

$$m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = \vec{a} \leftarrow \text{constant vector}$$

- integrate another time wrt time.

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = \vec{a} t + \vec{b}$$

defn. of center of mass: $\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

get $\dot{\vec{R}} = \frac{\vec{a}}{m_1 + m_2} = (\text{constant})$

$$\vec{R} = \frac{\vec{a} t}{m_1 + m_2} + \frac{\vec{b}}{m_1 + m_2}$$

The momentum of the 2-body system is conserved, and the center of mass moves with constant velocity.

→ knowing the motion of the system as a whole, all that is required is to know the motion of body 1 WRT body 2.

The equation of relative motion is:

$$\frac{d^2\vec{r}}{dt^2} + \mu \frac{\vec{r}}{r^3} = 0$$

Take the cross product:

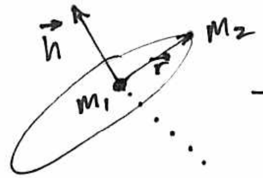
$$\vec{r} \times \ddot{\vec{r}} + \vec{r} + \frac{\mu \vec{r}}{r^3} = 0$$

indicates conservation of angular momentum ($\vec{h}$) + the fact that $\vec{r}$ and $\dot{\vec{r}}$ lie in a plane →

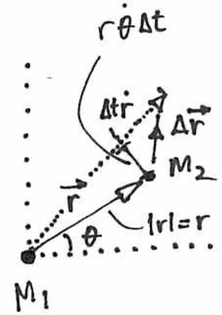
$$\vec{r} \times \ddot{\vec{r}} = 0$$

$\Big\downarrow$ (integrate)

$$\vec{r} \times \dot{\vec{r}} = \vec{h}$$



→ Use polar coords. →



in polar coordinates, the separation vector and its time derivatives are:

$$\vec{r} = r\hat{r}$$

(see diagram) →

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

[make sure you can derive this] →

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta})\hat{r} + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]\hat{\theta}$$

↓

Substitute this into the equation of relative motion

$$(\ddot{r} - r\dot{\theta})\hat{r} + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]\hat{\theta} + \mu\frac{r\hat{r}}{r^3} = 0$$

look at $\hat{r}$ componet...

(*) $\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$ → note $\vec{h} = \vec{r} \times \dot{\vec{r}}$
$|h| = r^2\dot{\theta}$

trick: $u = \frac{1}{r}$, use h conservation

+ differentiation of r using the chain rule

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta}$$

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Eqn (*) above can be re-written

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

- given two parameters $\mu = GM$ and $h = r^2\dot{\theta}$ this 2nd order dE tells how r varies with θ gives the orbital figure

This Eqn. is clearly related to simple harmonic motion – an orbit is an oscillation!

AMPLITUDE constant of integration

solution

$$u = \frac{\mu}{h^2}[1 + e\cos(\theta - \omega)]$$

constant of integration PHASE

recover r from $\frac{1}{u}$.

The constant P = "semilatus rectum" $= \frac{h^2}{\mu}$

$$r = \frac{P}{1 + e\cos(\theta - \omega)}$$

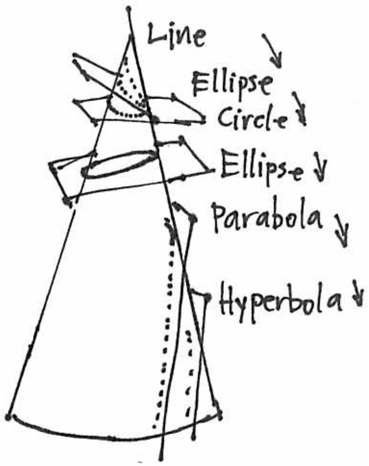
The general solution to the relative motion in the two body problem is the equation of a conic.

$$r = \frac{p}{1 + e\cos(\theta - \varpi)}$$

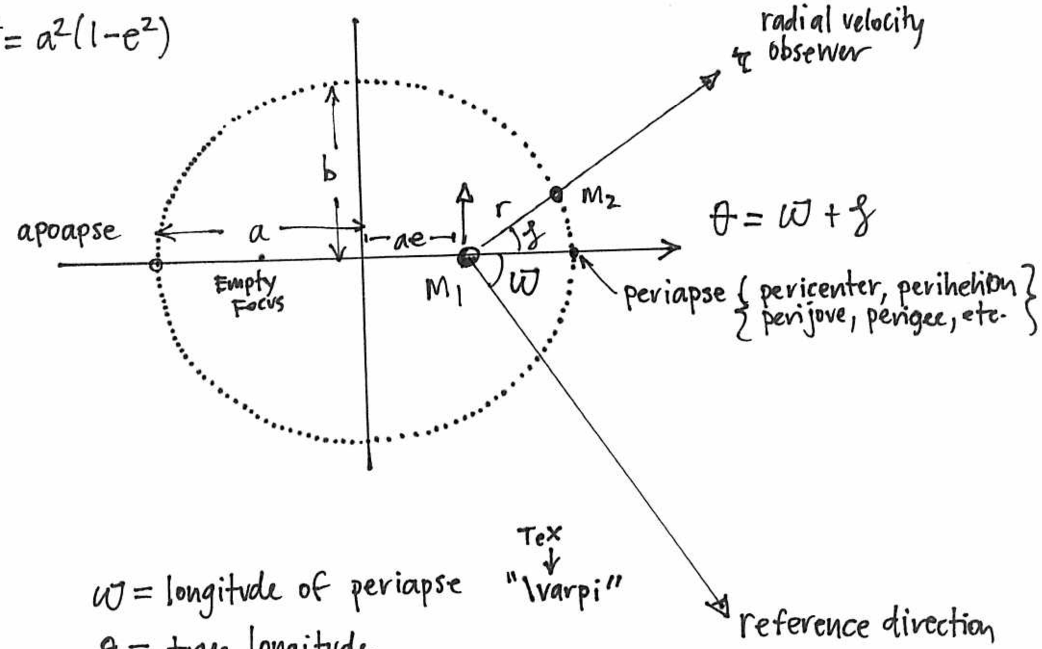
circle	$e = 0$	$p = a$
ellipse	$0 < e < 1$	$p = a(1-e^2)$
parabola	$e = 1$	$p = 2q$
hyperbola	$e > 1$	$p = a(e^2-1)$

a = semi major axis

q = distance at closest approach



$$b^2 = a^2(1-e^2)$$



ϖ = longitude of periapse   TeX → "\varpi"

θ = true longitude

f = true anomaly

close approach $d = a(1-e)$

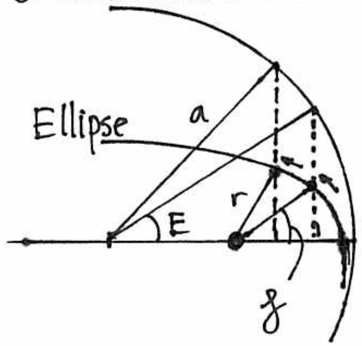
farthest away $d = a(1+e)$

so far we've been concerned with the orbital figure (which does not change in time). Where does the body spend its time on the figure?

define the Mean Anomaly: $M = \frac{2\pi}{P}(t - \tau)$

↑ time of periapse passage

The Mean anomaly is like a clock hand. It increases with steady linear accumulation. Unless the orbit is circular, it has no straightforward geometric interpretation.



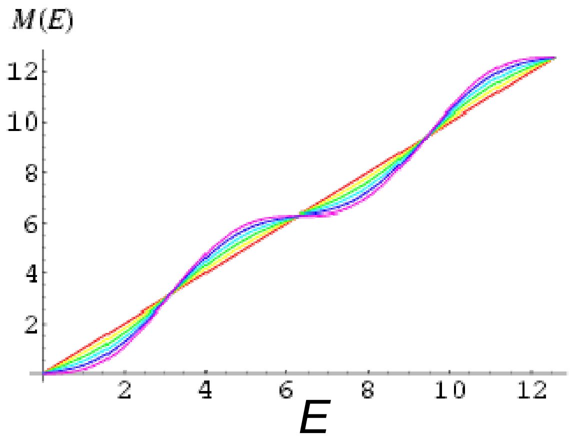
E = eccentric anomaly

can derive:

$$M = E - e\sin E$$

↑ relates "clock hand" to the position in the orbit

$M = E - e\sin E$ is Kepler's equation.
As far as I'm concerned, it is best solved numerically!



Note: for moderate eccentricities, one can get a quick hassle-free numerical solution via simple iteration:

$M_0 = E - e\sin E$

$M_1 = M_0 - e\sin M_0$

$M_2 = M_1 - e\sin M_1$

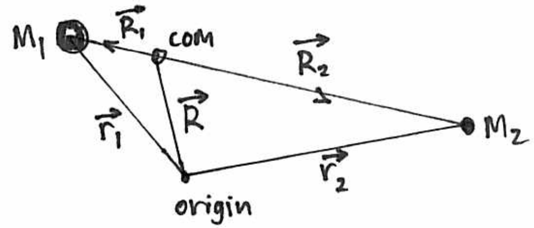
etc

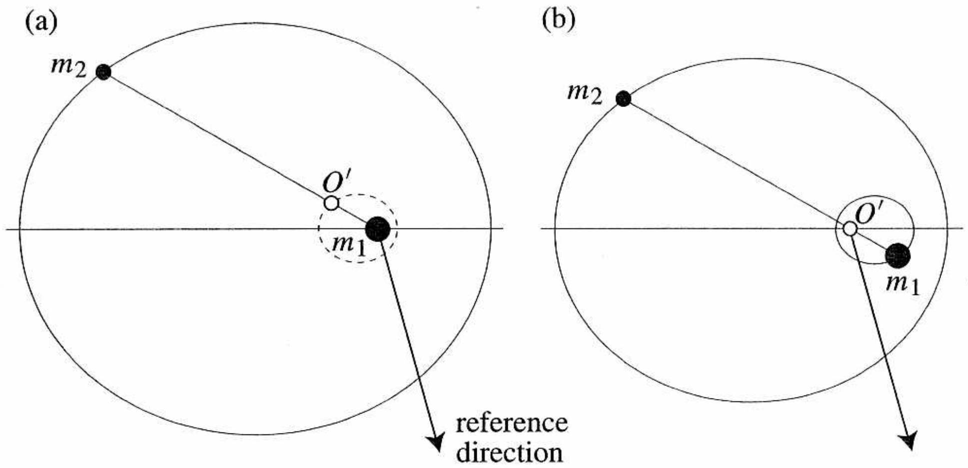
So far, we have been discussing the relative motion of the vector $\vec{r}$ that connects m_1 to m_2. Hence, from the point of view of either body, the other body is executing an elliptical orbit of semi-major axis a.

The center of mass is always on the line joining m_1 and m_2:

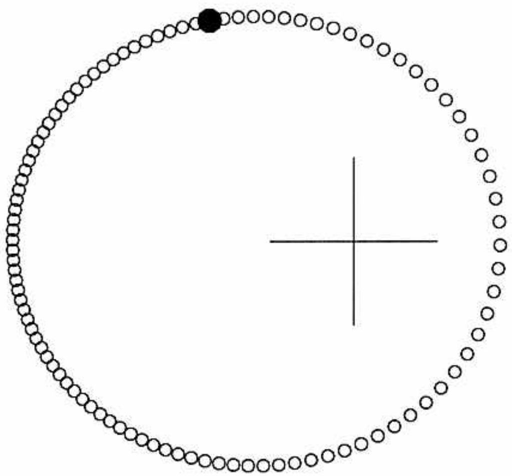
$$R_1 = \frac{m_2}{m_1+m_2} r \qquad R_2 = \frac{m_1}{m_1+m_2} r$$

Therefore, whichever conic section describes the relative motion of the two masses, each mass will also orbit the center of mass of the system in a path described by the same conic section reduced in scale by $m_1/(m_1+m_2)$ or $m_2/(m_1+m_2)$.

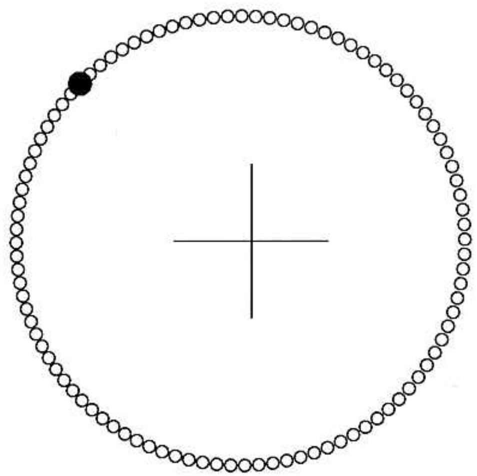




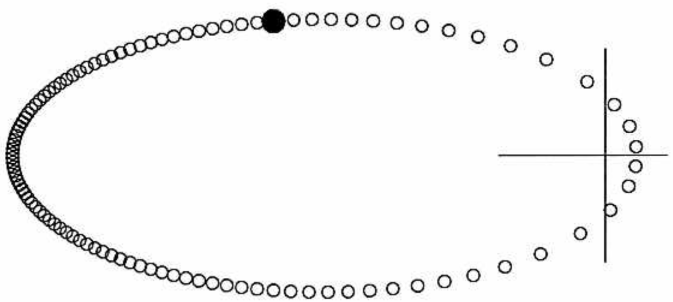
Eccentricity and Mean Anomaly Quiz



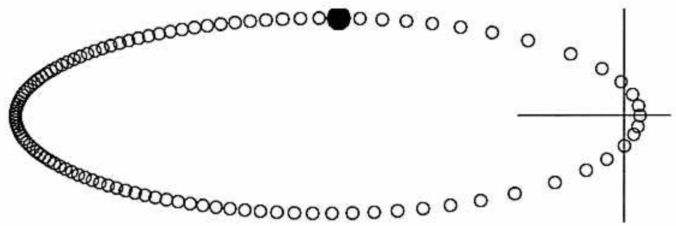
e =　　　M (deg) =



e =　　　M (deg) =

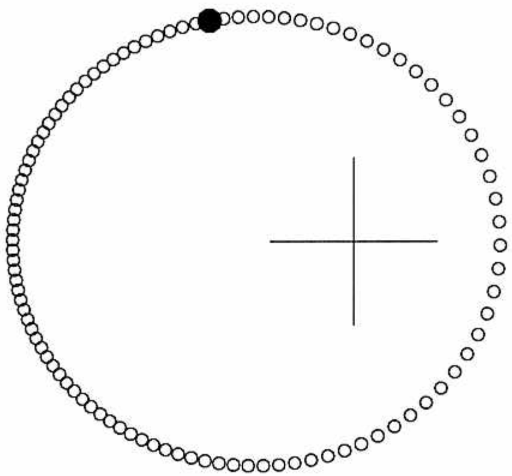


e =　　　M (deg) =

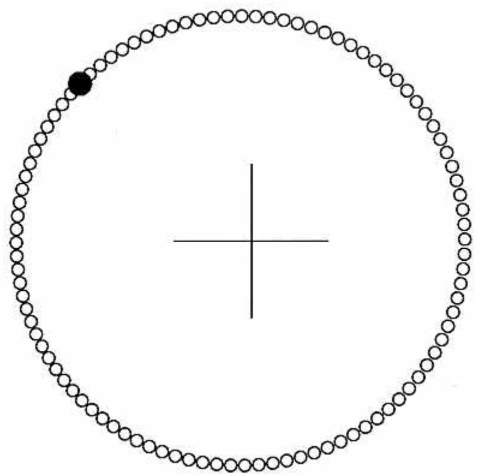


e =　　　M (deg) =

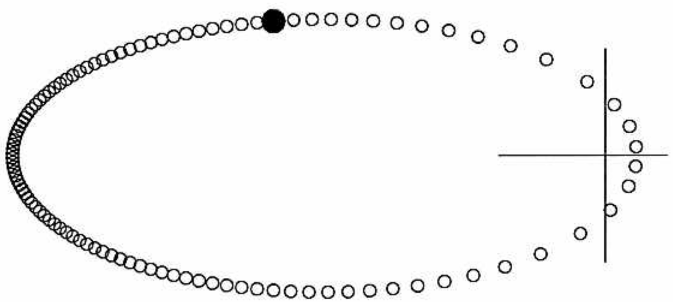
Eccentricity and Mean Anomaly Quiz



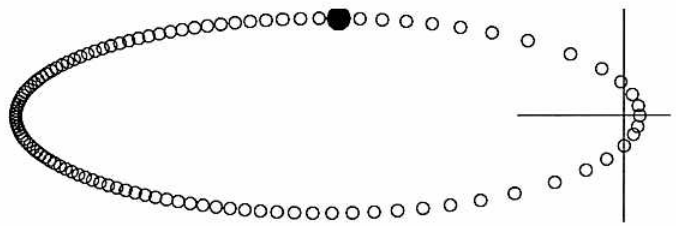
e = *0.4* M (deg) = *75*



e = *0.05* M (deg) = *130*



e = *0.9* M (deg) = *45*



e = *0.95* M (deg) = *30*

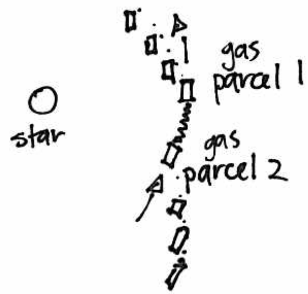
- An important example of the negative heat capacity property of the keplerian orbit is the "Magnetorotational instability" (e.g. Balbus & Hawley 376, 214, ApJ, 1991) originally discovered by Chandrasekhar.

↖ currently has 689 citations on ADS

↘ Chandrasekhar, S. 1960, Proc. Nat. Acad. Sci., 46, 53

In the "ideal MHD" regime, where the ionization fraction of the gas is high enough for the neutrals and ions to be coupled, magnetic field lines behave like rubber bands threading the gas.

TOP view



← imagine that gas parcel 1 is perturbed slightly backward

↓

This causes it to fall inward and speed up

↓

This causes increased tension from the connecting field line

↓

This causes further inward fall and speed up

↓

instability — feeds off the keplerian shear.

QuickTime™ and a YUV420 codec decompressor are needed to see this picture.

→ animation see "pnetp.mpg" on the links page ←

The Balbus Hawley instability likely provides an important source of angular momentum transport in protostellar disks.

- Talk about the negative heat capacity of the Keplerian orbit, and self gravitating systems in general
  - ↖ The KEY idea of astrophysics
    - stars
    - Magnetorotational Instability

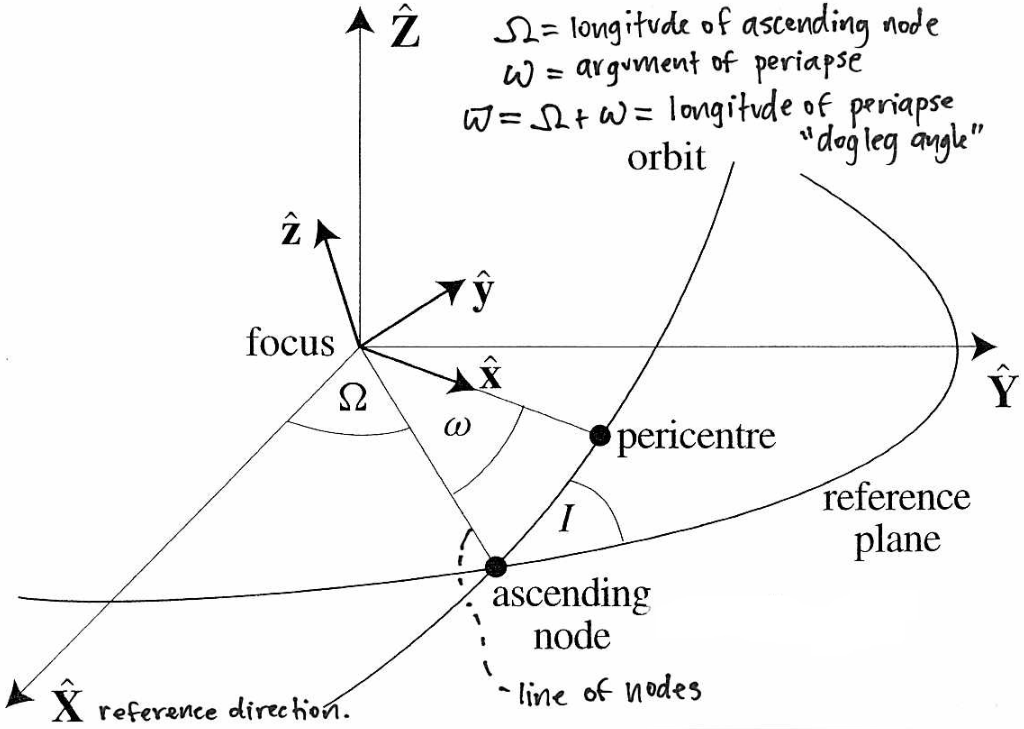
→ If we can't choose the plane to describe the 2·body motion, we need to deal with the orbit in 3D space:

I = inclination

Ω = longitude of ascending node

ω = argument of periapse

$\varpi = \Omega + \omega$ = longitude of periapse "dog leg angle"



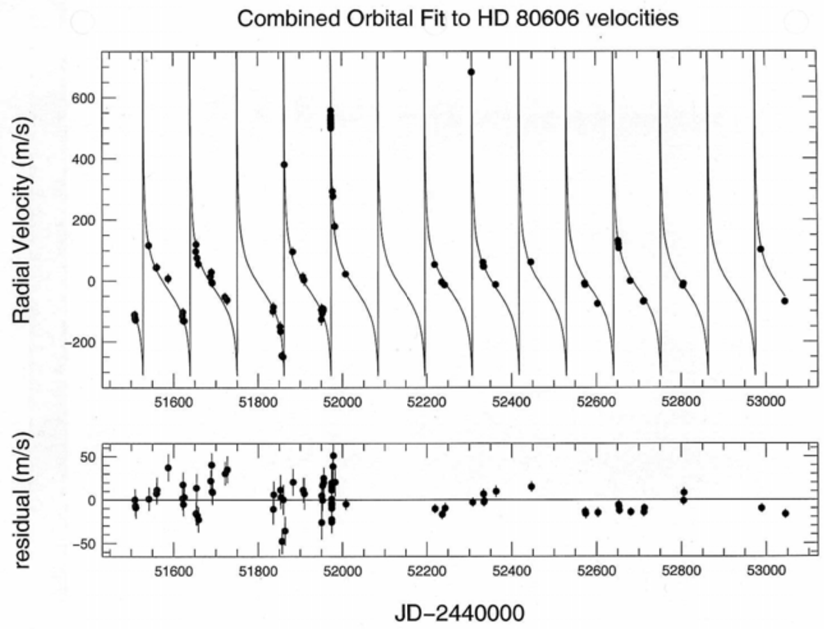
Orbital elements $a, e, M, \Omega, \omega, I$ are equivalent to cartesian $\vec{x}, \vec{v}$ x, y, z, v_x, v_y, v_z

↳ see routines in integrator.f.

M is the only non-conserved element...

Hi Everyone,

I've been looking at HD 80606 (the extremely high-e planet originally reported by the Swiss). About a month ago, Debra sent the 23 Keck velocities for the star (current through 245304.891). 55 Swiss velocities (with larger errors) have been published, so I have done some fits to the combined syster taking Keck-Elodie offset as a free parameter.



My best 1-planet fit to the combined system is pretty bizarre:

sqrt(chi^2)=1.62

P = 111.301003 d
Mean anomaly = 296.718994 deg at epoch JD 2451508.67
(equivalent to Tperi= JD 2451528.2416)
e = 0.9712 (!)
w = 309.946014
M=4.83 Mjup

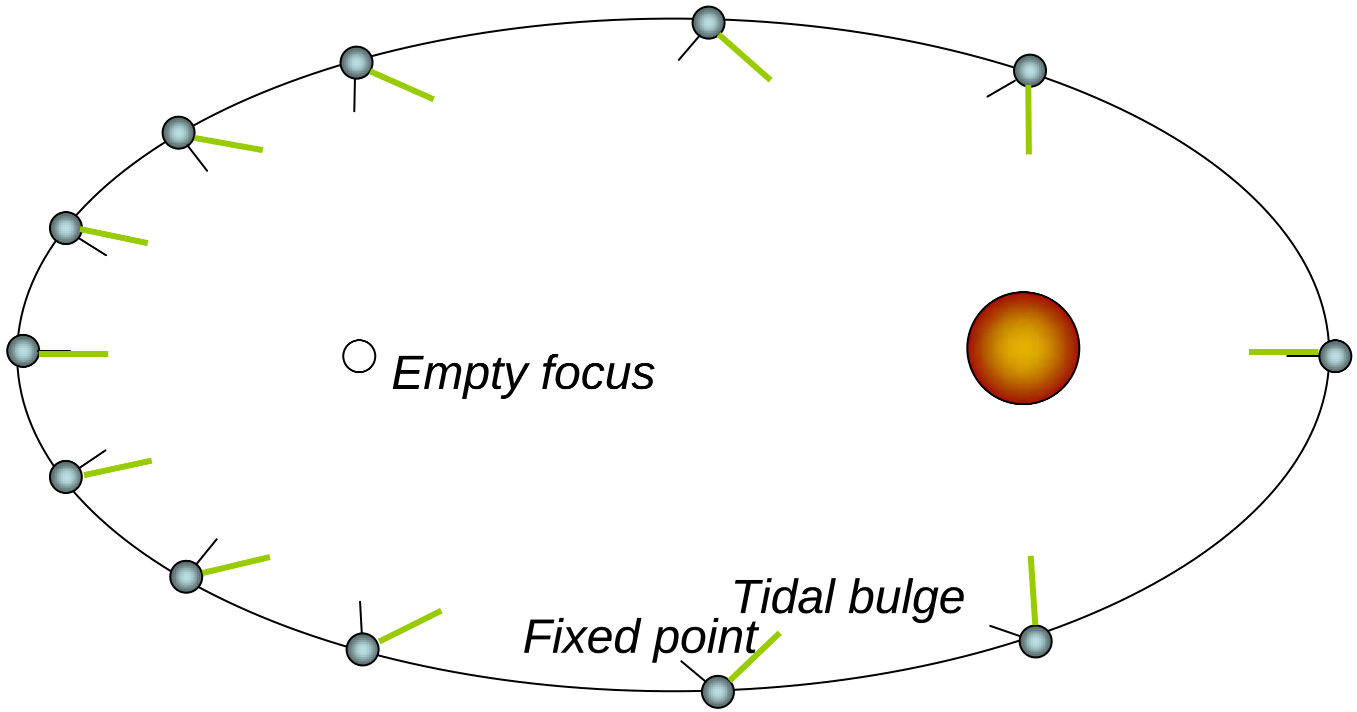
telescope offset = -85.82 m/s

The higher eccentricity is resulting from the following Keck pc
52307.873 682.42 4.85 & keck \\

As you can see from the attached RV plot, Keck caught the planet closer to perihelion than did the Swiss, showing that the radial velocity swing is greater than previously measured, which forced the eccentricity to a higher value. If this is confirmed, it would be a truly remarkable result.

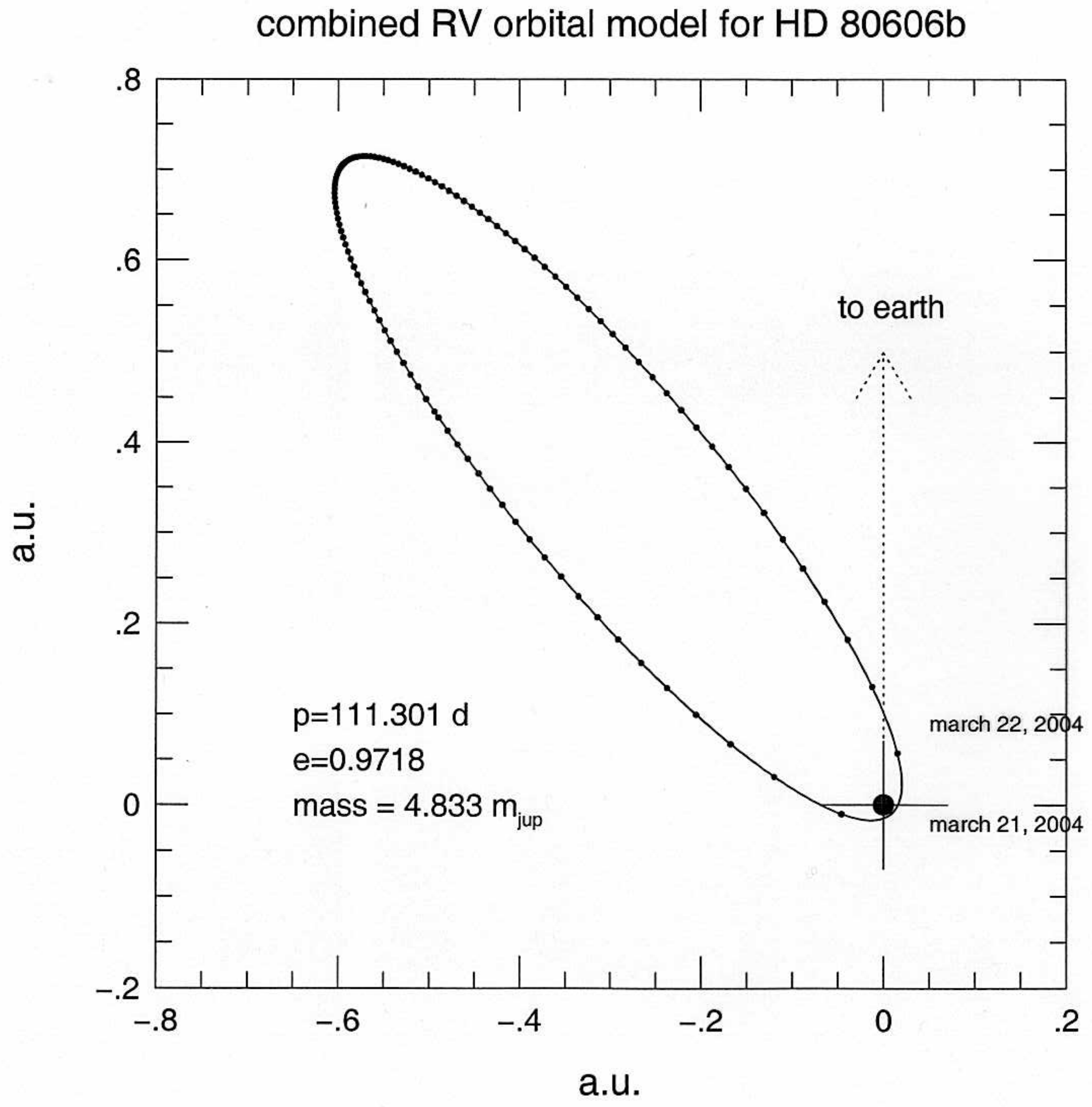
For the fit above, I compute a transit probability of 5%, assuming a 1 Rsun parent star. The transit is predicted for: JD 2453087.33 (March 22, 2004), and would last for about 9 hours.

A satellite with an eccentric orbit that is in synchronous rotation experiences Tidal Heating:



With it’s high eccentricity, HD 80606b is experiencing a lot of tidal heating. If e=0.97, the fact that the planet exists at all would put very important constraints on it’s structure. That is, it would have to have a very

QuickTime™ and a
GIF decompressor
are needed to see this picture.


combined RV orbital model for HD 80606b
.8
.6
.4
.2
0
–.2
–.8
–.6
–.4
–.2
0
.2
a.u.
a.u.
to earth
march 22, 2004
march 21, 2004
p=111.301 d
e=0.9718
mass = 4.833 m
jup

Hi Everyone,

Here is an update regarding the upcoming periastron passage of HD 80606b on July 11. It is a real stroke of luck that this extraordinary event corresponds with a Keck run.

First the airmass situation... HD 80606 is a serious stretch for July, and Geoff told me that the telescope operator will need to be notified in advance of the large hour angle involved.

HD 80606 airmass table for July 8-12 Keck run:

local time=20:00	Airmass	HA	Sun Alt	JD	
July 7	2.839	05 25	-12.4	2453194.75	
July 8	2.924	05 29	-12.4	2453195.75	
July 9	3.014	05 33	-12.4	2453196.75	
July 10	3.109	05 37	-12.5	2453197.75	<- critical night
July 11	3.211	05 41	-12.5	2453198.75	

Using the Monte Carlo generation of synthetic data sets method to estimate undertainties, I calculate the following fit to the combined Keck+Swiss data sets. Andrew, using an independent code, found a fit that is identical to within the estimated error bars.

P = 111.301 +/- 0.033 d
Mean anomaly = 296.7 +/- 0.8 deg at epoch JD 2451508.677
(equivalent to Tperi= JD 2453197.75)
-> e = 0.9712 +/- 0.018
w = 309.946014 +/- 5 deg
M=4.83 +/- 0.58 Mjup (assuming 1.1 Msun for the star)

The best fit therefore has the planet coming within 2 stellar radii at close approach, which would have many very interesting ramifications.

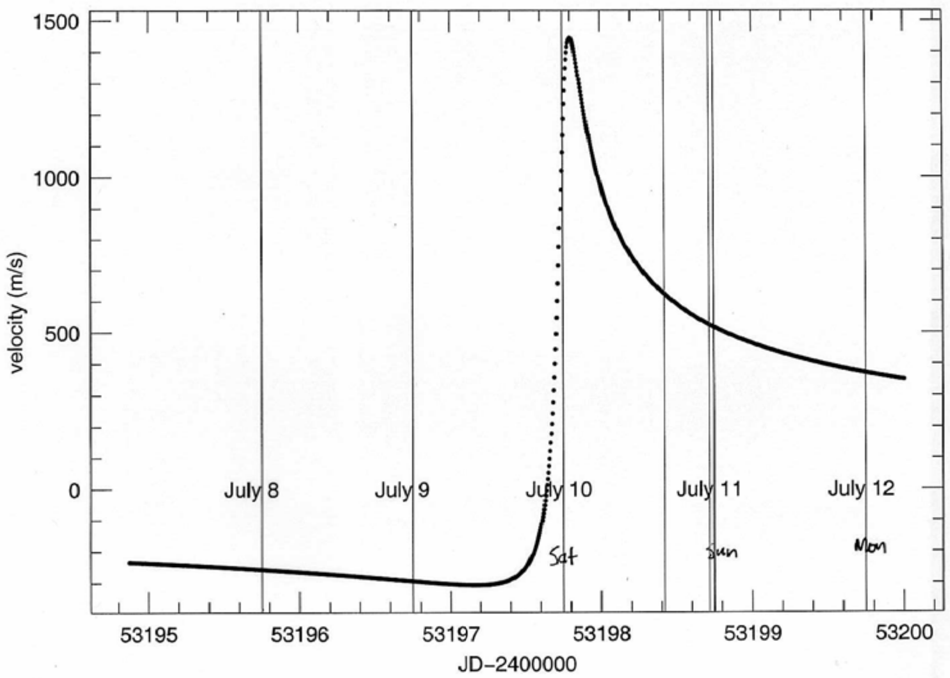
The attached postscript figure shows the predicted radial velocity curve during the upcoming Keck run. The four vertical red lines show 08:00 PM Keck time on July 9, July 10, July 11, and July 12. For the best-fit to the current data, the July 10th observing opportunity falls right on the rapidly varying portion of the radial velocity curve. RV's obtained on this portion of the curve will strongly constrain the eccentricity. Currently, the location of the big swing is uncertain by about 1/3 of a day.

The two green lines show the predicted ingress and egress times in the event of a transit. The July 11 opportunity falls very close to this window, and gives rise to the possibility of measuring the Rossiter McLaughlin effect. The transit probability is about 5%.

Here is the predicted radial velocity curve for the early evening of July 10th.

JD-2400000 vel (m/s)

53197.6992	446.079922	
53197.7033	494.890967	
53197.7075	546.391175	
53197.7117	600.456767	
53197.7158	656.876289	
53197.7200	715.337547	
53197.7242	775.418108	
53197.7283	836.581773	7:30 PM local time (Mauna Kea)
53197.7325	898.183490	
53197.7367	959.484635	
53197.7408	1019.67955	
53197.7450	1077.9324	
53197.7492	1133.42161	8:00 PM (air mass=3.109)
53197.7533	1185.38693	
53197.7575	1233.17368	
53197.7617	1276.26817	
53197.7658	1314.32025	
53197.7700	1347.15077	8:30 PM
53197.7742	1374.74463	
53197.7783	1397.23191	
53197.7825	1414.86115	
53197.7867	1427.96903	
53197.7908	1436.95018	
53197.7950	1442.22991	
53197.7992	1444.24142	
53197.8033	1443.40831	
53197.8075	1440.13202	
53197.8117	1434.78379	



The velocity swing is really quite extraordinary. A *six minute* exposure started at 07:30 PM on July 10th will span a reflex velocity change of 60 m/s. I think the best strategy on the 11th would therefore be to take one exposure of the usual length (to get precision in the event that the big swing occurs several hours earlier or later) and then at least one short exposure to hit the sweet spot between maximizing spectral S/N, and minimizing velocity drift during the exposure itself. Also, if the July 9th and 10th points are reduced prior to the evening of the 10th, we would have a much better idea of when on the 11th the big swing is going to occur.

best,
Greg

Hi all,

All is well with HD 80606.

In the bag are observations of HD 80606 from 6 Keck nights, July 2, 3, 8, 9, 10, 11 .
There is still one more night of Keck data to process, from monday night (July 12/13)
which was taken at JD = 24513199.74 .

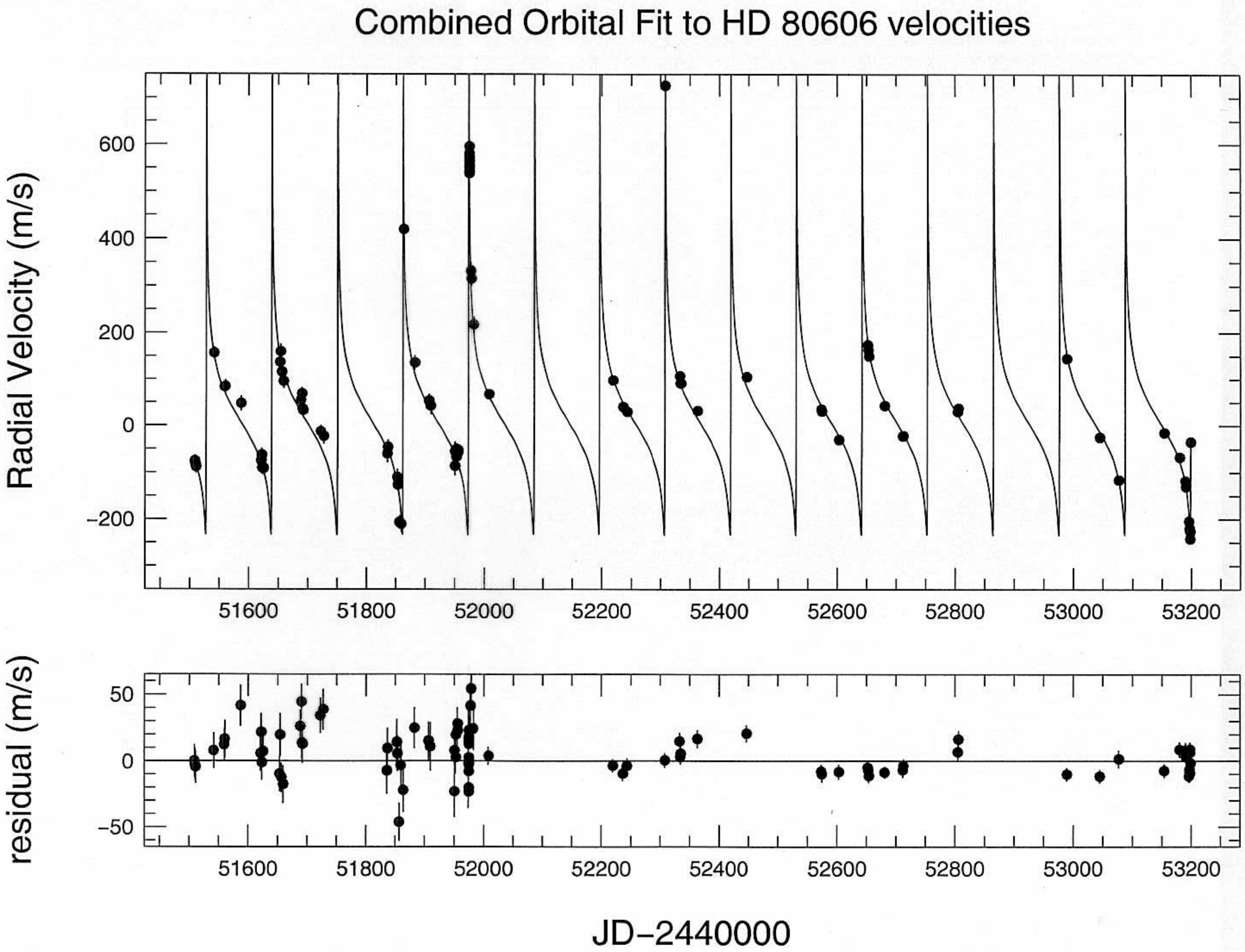
You can fit all extant velocities with a simple Keplerian, with no slope, and ecc = 0.945 .

On July 11, the velocities rose 190 m/s from the previous night, so the very last night of observation,
yet to be processed, might show velocities near the peak (again).

The RMS = 9.8 m/s, is a bit high, as internal errors are ~5 m/s and jitter is expected
to be 2.6 m/s.

These were tricky observations, as the star was at Hour Angle, HA = 5 hr West,
setting into the 12 degree twilight. Paul is working on the Doppler analysis of the last night.
The extant velocities are listed below.

Onward,
Geoff



Combined Orbital Fit to HD 80606 velocities
Radial Velocity (m/s)
600
400
200
0
−200
51600
51800
52000
52200
52400
52600
52800
53000
53200
residual (m/s)
50
0
−50
JD−2440000

Radial Velocity Fitting Applet

www.ucolick.org/~laugh/SystemicBeta/websystemic.html

NUMERICAL INTEGRATION

The numerical integration of orbits.

Given a second-order ODE: (or higher order)

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

y: dependant variable; x: independant variable

recast it as two (or more) first-order equations:

$$\frac{dy}{dx} = z(x)$$

$$\frac{dz}{dx} = r(x) - q(x)\,z(x)$$

Problems in higher order ordinary differential equations therefore reduce to coupled sets of first order ODE's

$$\frac{dy_i(x)}{dx} = f_i(x, y_1, y_2, \ldots, y_n)$$

How these equations are attacked depends on the form of the boundary conditions (1 condition per equation)

The gravitational N-body problem

above here: $y \leftrightarrow x$, $x \leftrightarrow t$

$$\frac{d^2\vec{x}_i}{dt^2} = -\sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|^2} = f(\vec{x}_1, \ldots, \vec{x}_i, \ldots \vec{x}_n)$$

note no t dependance on RHS

• depends on conditions at $t=0$...

The N-body problem is an "initial value problem" →

$$\frac{d\vec{x}_i}{dt} = \vec{v}_i$$

$$\frac{d\vec{v}_i}{dt} = -f(\vec{x}_1, \ldots, \vec{x}_i, \ldots, \vec{x}_n)$$

← RHS's given at t=0.

} Right hand sides computed in subroutine "derivs"

see → fewbody.f, integrator.f

The simplest possible N-body code consists of Euler's method for differencing these equations:

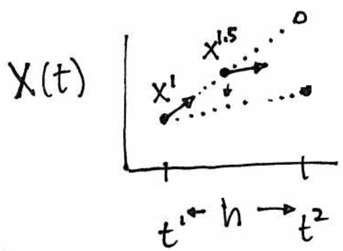
$$x_i^{n+1} = x_i^n + (t^{n+1} - t^n)\, f(t^n, x_1^n, \ldots, x_i^n, \ldots, x_N^n)$$

Error in this accumulates as $\mathcal{O}(h^2)$ →

↑ $t^n \to t^n + \Delta t = t^{n+1}$

if you are new to numerical work, try coding up the simplest possible N-body code using Euler's method.

Consider a simple refinement to Euler's method: the Midpoint Method



Use Euler's method to step to the middle of the interval.

This gives $X^{1.5}$, and hence the opportunity to calculate a new estimate for the derivative:

$$\frac{dx^{1.5}}{dt^{1.5}} = f(\vec{x}^{1.5})$$

Then use this midpoint derivative to step across the entire interval ←

↓ Error terms accumulate as $\mathcal{O}(h^3)$

↘ Equivalent to the trapezoidal rule

The numerical algorithm for the modified midpoint method is

← t present in general, but not in n-body

$$k_1 = h\, f(t', x')$$

$$k_2 = f(t^{1.5}, x' + \frac{h}{2} k_1)$$

$$x^2 = x' + h k_2$$

There are many ways to estimate the proper value $f(t,x)$ that correctly weights $f(t,x)$ across the entire interval. These have different coefficients of higher-order terms. Add up fractions of the different $f(t,x)$ estimates so that the error terms cancel out to the desired order

* higher order is not always higher accuracy *

↳ but for smooth functions it tends to be.

The classical 4th-order Runga Kutta formula is the most often used for numerical integration

$$k_1 = f(t', x')$$

$$k_2 = f(t^{1.5}, x' + \frac{h}{2} k_1)$$

$$k_3 = f(t^{1.5}, x' + \frac{h}{2} k_2)$$

$$k_4 = f(t^2, x' + h k_3)$$

Think of the k's as slopes. →

$$X^2 = X' + \frac{h}{6} k_1 + \frac{h}{3} k_2 + \frac{h}{3} k_3 + \frac{h}{6} k_4$$

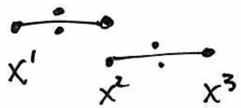
For the N-body problem, we have no time dependance in the force law, so energy (readily computed in cartesian coordinates) is conserved:

$$E = \frac{1}{2} m V_{com}^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} -\frac{G m_i m_j}{r_{ij}} + \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

$\Delta E = E_{final} - E_{initial}$ provides an estimate of whether the timestep is ok.

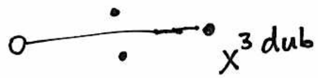
Step Doubling: "ridin' on dubs"

1. Two small steps using Runga-kutta



→ Eight slope estimates (derivs calls)

2. One big step of twice the size:



This requires three more slope estimates, since the first one was done at high resolution.

Overhead = 1.375

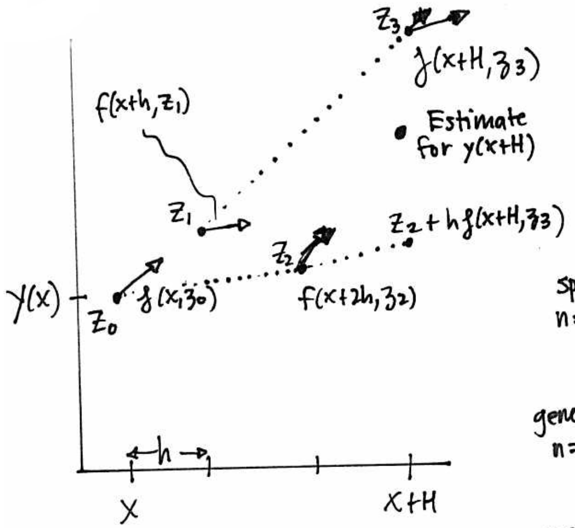
At every two steps get quantity $\Delta = x^3 - x^3_{dub}$ = Error Estimate.

This needs to stay under a desired criterion $\Delta_{tolerated}$. Varying h allows you to meet $\Delta_{tolerated}$.

"Adaptive stepsize control" { Δ too big? decrease h and restart; Δ too small? increase h for next step

The Bulirsch-Stoer Method
solutions with a minimum effort...

Break x to $x+H$ into n subintervals: $h = \frac{H}{n}$



$z_0 = y(x)$ = known at start

$$z_1 = z_0 + h\,g(x, z_0)$$

specific $n=3$:
$$z_2 = z_0 + 2h\,f(x+h, z_1)$$
$$z_3 = z_1 + 2h\,f(x+2h, z_2)$$

general $n=m$:
$$z_{m+1} = z_{m-1} + 2h\,g(x+mh, z_m)$$
for $m = 1, 2, n-1$

specific $n=3$:
$$y(x+H) \simeq \frac{1}{2} z_3 + \frac{1}{2}\left(z_2 + h\,g(x+H, z_3)\right)$$

general $n=m$:
$$y(x+H) \simeq \frac{1}{2}[z_n] + \frac{1}{2}[z_{n-1} + h\,g(x+H, z_n)]$$

↑ "modification"

(The midpoint method would just use $y(x+H) \simeq z_3$)

It turns out that the error of the modified midpoint method is second-order accurate, but the power series describing the error contains only even powers of the subinterval h:

$$y_n - y(x+H) = \sum_{i=1}^{\infty} \alpha_i h^{2i}$$

y_n ↑ numerical estimate; $y(x+H)$ ↖ true value

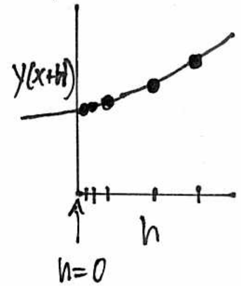
- By combining estimates, can gain 2 orders of accuracy at a time:

For example, if n is even, let $y_{n/2}$ be the result of applying the modified midpoint method with $n/2$ steps and y_n be the result of applying it with n steps.

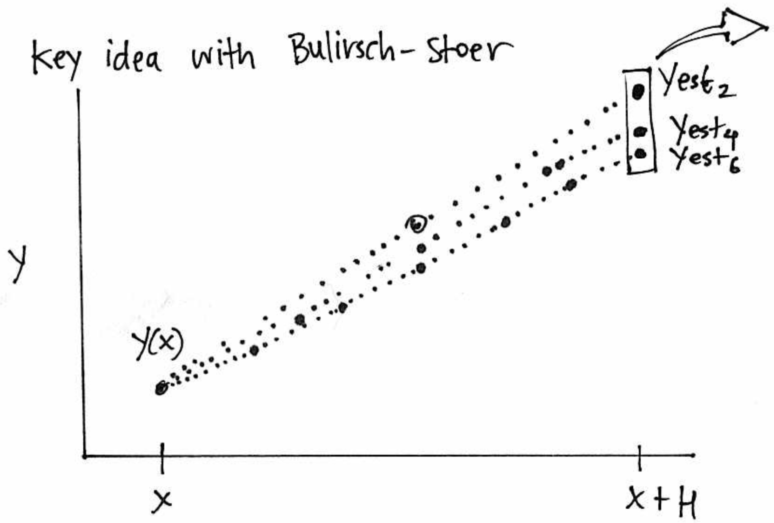
Then $$y(x+H) \simeq \frac{4y_n - y_{n/2}}{3}$$

turns out to be 4th order accurate.

but requires 1.5 derivative evaluations per step h, as opposed to 4 for Runge Kutta.



Key idea with Bulirsch-Stoer



1. Do the modified midpoint method with 2 steps
2. Do mmm w/ 4, 6, 8, 10, 12, 14... n_{max}
3. Fit a polynomial to $y(x+H)(h)$ and extrapolate it through $h=0$ → great estimate.
4. monitor $y(x+H)_{new} - y(x+H)_{old}$ until < criterion
5. stop when $n = n_{max}$ $\overline{y(x+H)_{new}}$

The Bulirsch-Stoer method is implemented for the planetary N-body problem using:

`integrator.f` (on the course website)

Detection of a NEPTUNE-mass planet in the ρ^1 Cancri system using the Hobby-Eberly Telescope

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Debra A. Fischer[2],Geoffrey W. Marcy[2,3], and R. Paul Butler[4]

Dominique Naef[5,6], Michel Mayor[5], Diedre Queloz[5], and Stephane Udry[5]

and

Thomas E. Harrison[7]

ABSTRACT

We report the detection of the lowest mass extra-solar planet yet found around a Sun-like star - a planet with an $M \sin i$ of only 14.21 ± 2.91 Earth masses in an extremely short period orbit (P=2.808 days) around ρ^1 Cancri , a planetary system which already has three known planets. Velocities taken from late 2003-2004 at McDonald Observatory with the Hobby-Eberly Telescope (HET) revealed this inner planet at 0.04 AU. We estimate an inclination of the outer planet ρ^1 Cancri d, based upon *Hubble Space Telescope* Fine Guidance Sensor (FGS) measurements which suggests an inner planet of only 17.7 ± 5.57 Earth masses, if coplanarity is assumed for the system.

Subject headings: (stars:) planetary systems — stars:individual (ρ^1 Cancri — astrometry

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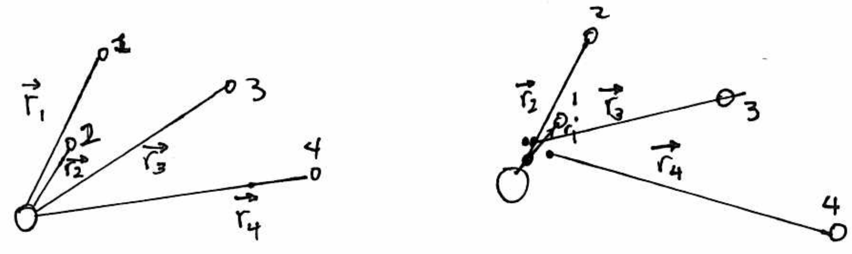
"And hence, if several lesser bodies revolve about a greatest one, it can be found that the orbits described will approach closer to elliptical orbits, and the description of errors will become more uniform [] if the focus of each orbit is located in the common center of gravity of all the inner bodies."

Newton, Book I., Section II, proposition 69, Principia



Astrocentric

Jacobi

(see attached paper)

Radial velocity fits are best considered as "osculating" Jacobi coordinates.

from the latin "osculare", to kiss

But which epoch should we use?

$$M_{epoch} = \left[\frac{T_{epoch} - T_{peri}}{Period} \right] \cdot 360^\circ$$

1. choose an epoch
2. Integrate the 55 cancri system
3. what happens?

Orbital Parameters to integrate:

Table 2. Quad-Keplerian Orbital Elements of ρ^1 Cancri

Element	ρ^1 Cancri e	ρ^1 Cancri b	ρ^1 Cancri c	ρ^1 Cancri d
Orbital Period P (days)	2.808 ± 0.002	14.67 ± 0.01	43.93 ± 0.25	4517.4 ± 77.8
Epoch of Periastron T [a]	3295.31 ± 0.32	3021.08 ± 0.01	3028.63 ± 0.25	2837.69 ± 68.87
Eccentricity e	0.174 ± 0.127	0.0197 ± 0.012	0.44 ± 0.08	0.327 ± 0.28
ω (°)	261.65 ± 41.14	131.49 ± 33.27	244.39 ± 10.65	234.73 ± 6.74
Velocity amplitude K (m s^{-1})	6.665 ± 0.81	67.365 ± 0.82	12.946 ± 0.86	49.786 ± 1.53
V_0 Lick (m s^{-1})	21.166 ± 1.31			
V_0 ELODIE (m s^{-1})	2727.448 ± 2.42			
V_0 HET (m s^{-1})	10.745 ± 0.59			

[a] Add 2450000.0 to T

→ You choose the epoch!

→ Is the system stable?

Table 3. ρ^1 Cancri - Mass Limits and Parameters

Parameter	ρ^1 Cancri e	ρ^1 Cancri b	ρ^1 Cancri c	ρ^1 Cancri d
a (AU)	0.038 ± 0.001	0.115 ± 0.003	0.240 ± 0.008	5.257 ± 0.208
$A \sin i$ (AU)	1.694e-6 $\pm$ 0.19e-6	9.080e-5 $\pm$ 0.12e-5	4 .695e-5 $\pm$ 0.14e-5	0.195e-1 $\pm$ 0.007e-1
Mass Fraction ($M_\odot$)	8.225e-14 $\pm$ 2.33e-14	4.64e-10 $\pm$ 0.17e-10	7.151e-12 $\pm$ 0.54e-12	4.874e-08 $\pm$ 0.38e-8
$M \sin i$ (M_{JUP})[a]	0.045 ± 0.01	0.784 ± 0.09	0.217 ± 0.04	3.912 ± 0.52
$M \sin i$ ($M_{NEPTUNE}$)[a]	0.824 ± 0.17			
$M \sin i$ (M_{EARTH})[a]	14.210 ± 2.95			
M (M_{JUP})[b,d]	0.056 ± 0.017	0.982 ± 0.19	0.272 ± 0.07	4.9 ± 1.1
M (M_{JUP})[c,d]	0.053 ± 0.020	0.982 ± 0.26	0.244 ± 0.07	4.64 ± 1.3
M ($M_{NEPTUNE}$)[c,d]	1.031 ± 0.34			
M (M_{EARTH})[c,d]	17.770 ± 5.57			

[a]derived from radial velocity alone

[b]derived from radial velocity and astrometry, using Msini/sini

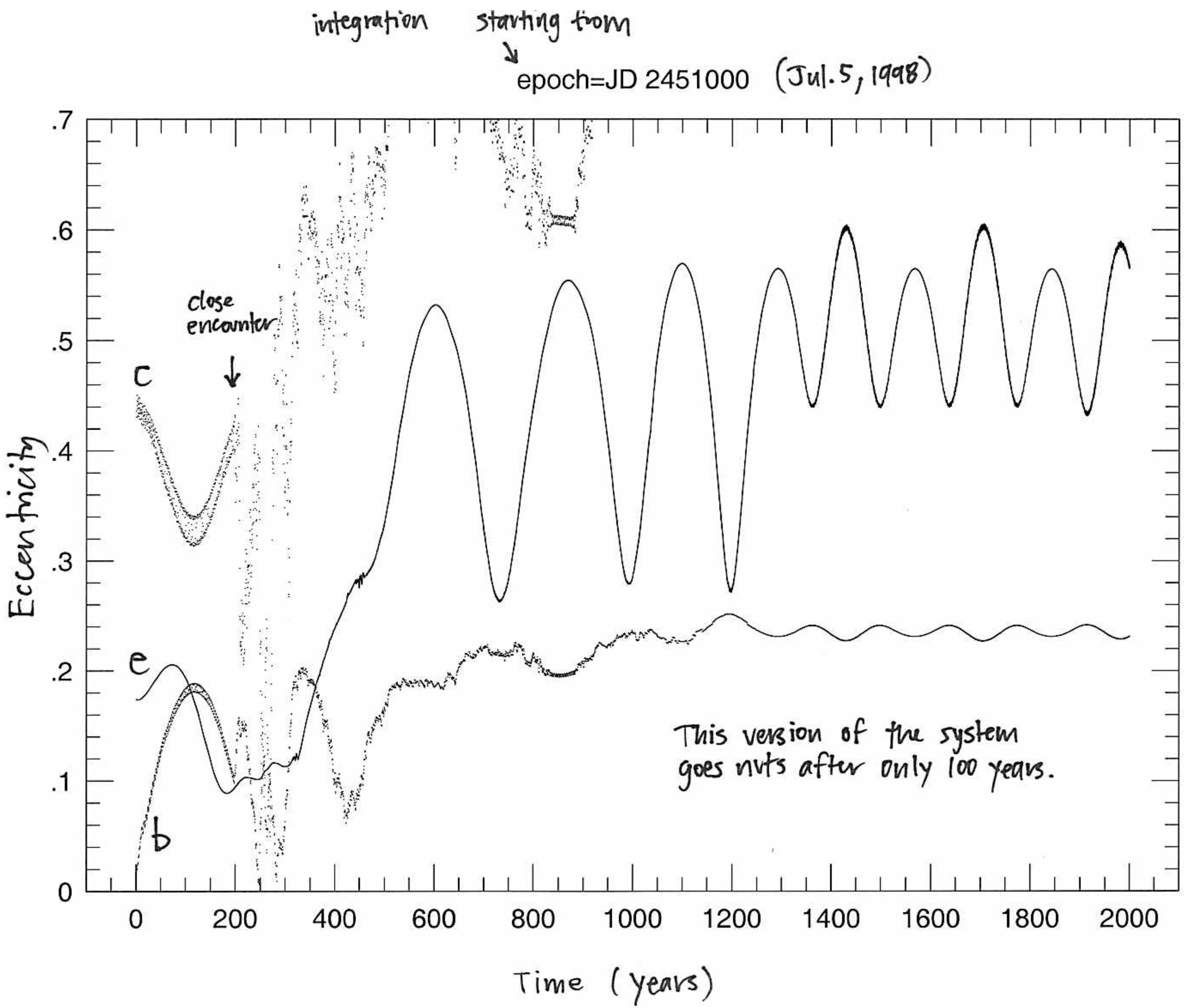
[c]derived from radial velocity and astrometry, using $m2^3/(m1+m2)^2 = a^3/P^2$

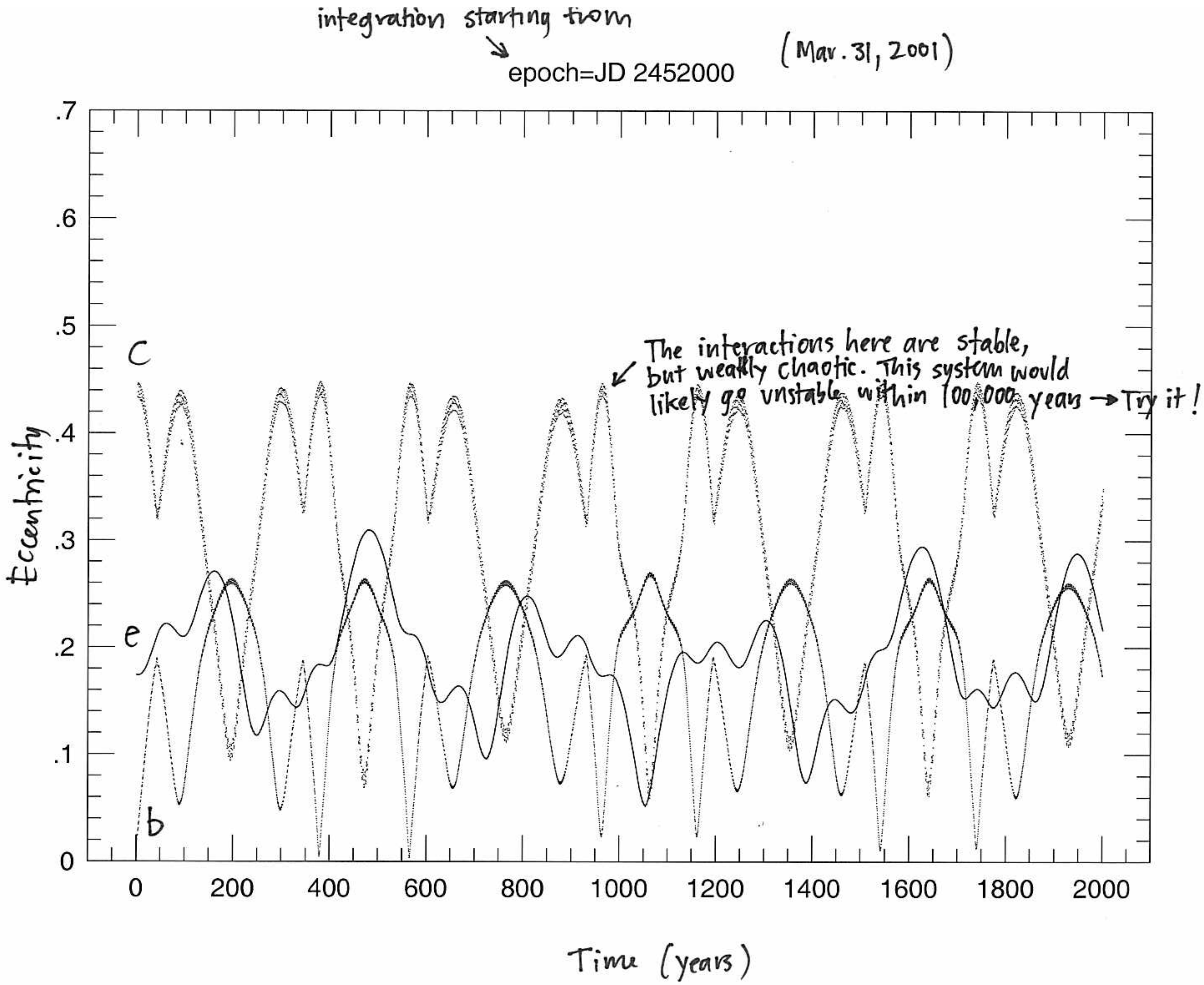
[d]assumes coplanarity of the planetary system

4-planet version of the 55 cancri system:

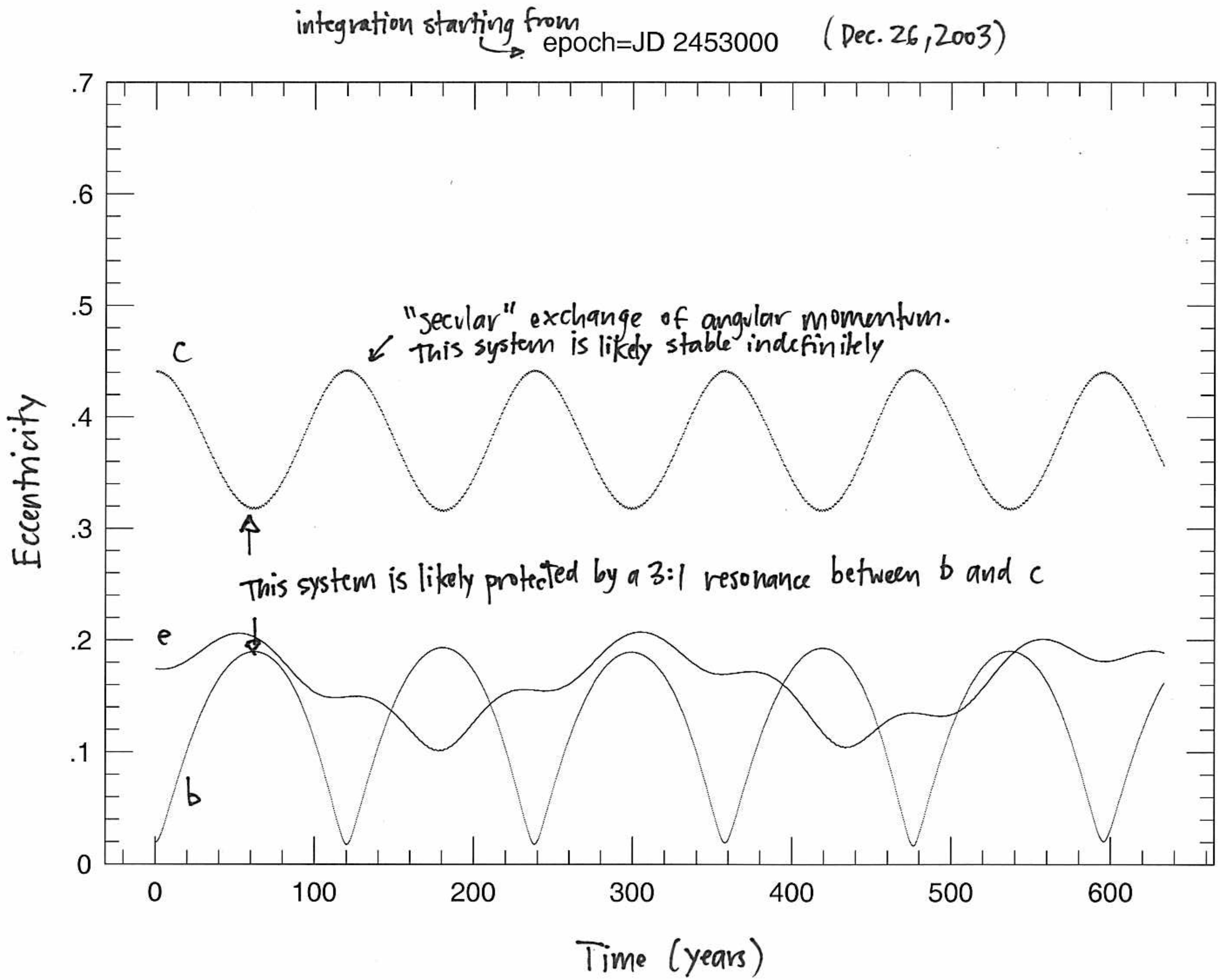
```
2000.        time of integration (years)
500          number of timesteps per print (watch out!)
0.95         mass of the central star (solar masses)
0            set to 1 if making a surface of section (nb should be 3)
1            set to 0 if reading in semi-major axes, 1 if reading periods
2.808        periods (days) or semi-major axes (au)
14.67
43.93
4517.4
1            set to 0 if reading m anom, 1 if Time of Peri, 2 if m longitude
3295.31      mean anoms (deg), or Peri Passages (days), or m longitudes (deg)
3021.08
3028.63
2837.69
3000.00      Starting Epoch (JDs) (both Tperi and Epoch are D-2450000)
0.174        eccentricities
0.0197
0.44
0.327
261.65       longitudes of pericentre (deg)
131.49
244.39
234.73
0.000        inclinations (deg) (This is 90-i, as usually defined)
0.000
0.000
0.000
0.000        longitudes of ascending node (deg)
0.000
0.000
0.000
0            set to 1 if inputting mass, 0 if inputting r.v. half-amp
6.665        Masses (x10^27 g)
67.365       ← Radial velocity half-amplitudes
12.946
49.786
1.0e-13      individual timestep accuracy for bulirsch-stoer
0.10         timestep interval for integrator (fraction period 1)
1            set to 0 for astrocentric coords, 1 for jacobi coords
```

the motion → (3000.00)
ere really the
um of keplerian
ien it should
ot matter
hich epoch
ou choose to
tart the
ntegrations.
↓
y picking a
articular
poch, we are
aking a choice
f a set of
"osculating"
nitial condition


integration starting from
epoch=JD 2451000 (Jul. 5, 1998)
close encounter
c
e
b
.7
.6
.5
.4
.3
.2
.1
0
Eccentricity
0
200
400
600
800
1000
1200
1400
1600
1800
2000
Time (years)
This version of the system goes nuts after only 100 years.

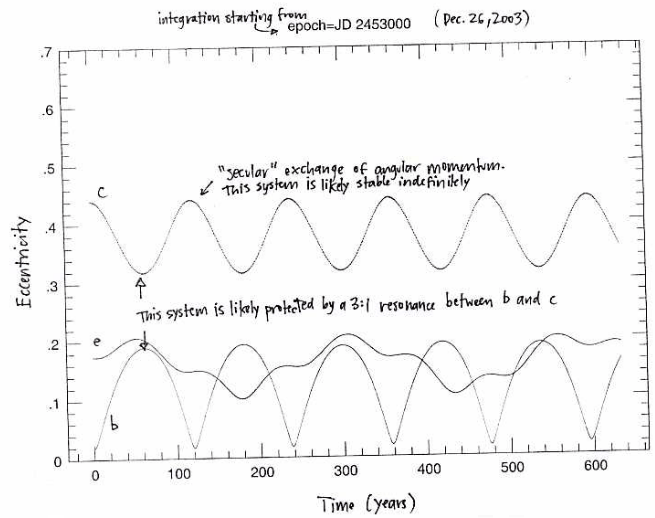


integration starting from
epoch=JD 2452000
(Mar. 31, 2001)
The interactions here are stable, but weakly chaotic. This system would likely go unstable within 100,000 years → Try it!
c
e
b
Eccentricity
.7
.6
.5
.4
.3
.2
.1
0
0
200
400
600
800
1000
1200
1400
1600
1800
2000
Time (years)



integration starting from
epoch=JD 2453000
(Dec. 26, 2003)
.7
.6
.5
.4
.3
.2
.1
0
Eccentricity
c
"secular" exchange of angular momentum.
This system is likely stable indefinitely
This system is likely protected by a 3:1 resonance between b and c
e
b
0
100
200
300
400
500
600
Time (years)

… Motions of the planets – secular variations, resonances, and the stability of planetary systems …



← Understanding what is happening in 55 Cancri is obviously a matter of current importance in astronomy

The great triumphs of the 18th century enlightenment have found a sudden resurgance in vitality with the discovery of extrasolar planets. Laplace, Euler, and Lagrange could be producing useful work within 10 minutes were they to arrive on the scene today.

1625 : Kepler remarks that Jupiter and Saturn do not behave quite in accordance with his empirically determined laws.

Newton couldn't explain this. → (He offered a non-secular solution)

"Secular Motion":

- For Jupiter, $n = \frac{2\pi}{P_J}$ is too small → period always too large, relative to later obs. → Jupiter is spiraling inward
- For Saturn $n = \frac{2\pi}{P_S}$ is too large → period too small, relative to later obs. → Saturn is spiraling outward

Halley (early 1700's) finds that in 2000 years, the acceleration of Jupiter amounts to $3° 49'$ along the ecliptic, while the acceleration of Saturn amounts to $-9° 16'$.

↳ This discrepancy became known to the continental mathematicians who picked up where Newton left off as "Le Grande Inegalite" – The Great Inequality.

1748: Paris Academy offers a prize for the explanation

1752: Euler develops the framework

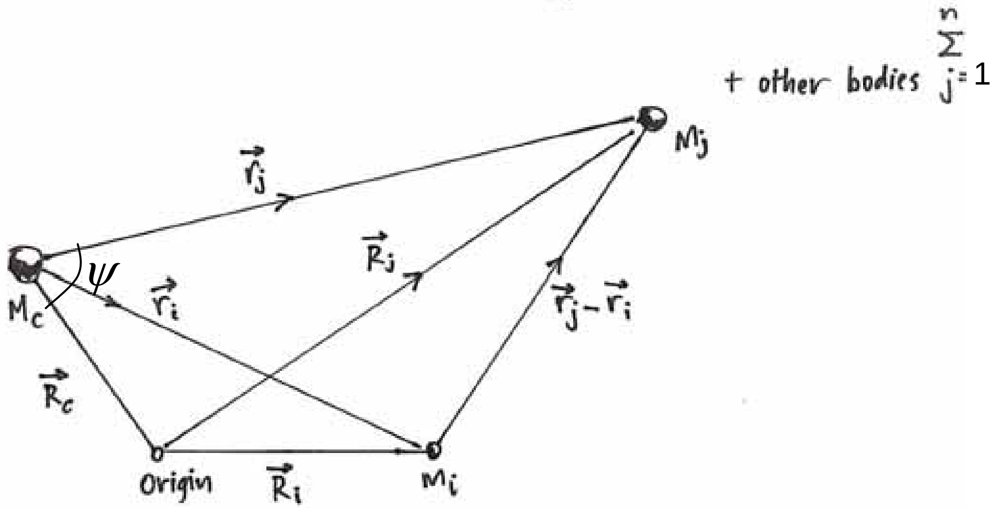
1) Imagines perturbations from one planet onto another as changes to the (nearly) conserved elements of an ellipse: $a, e, I, \varpi, \Omega, M$

2) "Euler computed the motion wholly by the elliptic theory, upon the supposition that the planet continually revolved in an ellipse, the elements of which varied every instant from the action of the other planets"

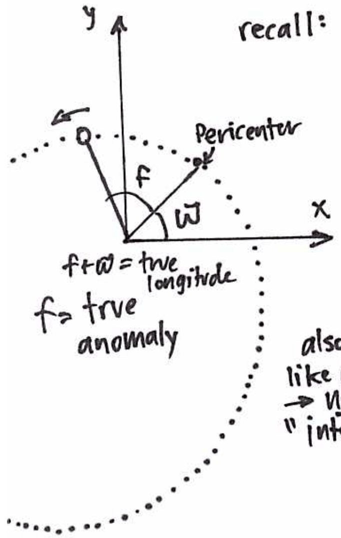
↳ Solve a (set of) differential equations for the instantaneous values of the elements as a function of time.

↳ with the instantaneous values of the elements, you immediately <u>know</u> the position of the planet.

The Disturbing Function — "It's disturbing." –D.N.C. Lin



- M_i and M_j are both in near-keplerian orbits around M_c
- orbital motion for M_i described (closely) by $a_i, e_i, I_i, \Omega_i, \varpi_i, \lambda_i$



recall: $M = n(t-\tau)$ → increases like a clock hand.

- τ: time of pericenter passage
- n: $2\pi/P$
- M: mean anomaly

mean longitude

$\lambda = M + \varpi$

- ϖ: longitude of pericenter
- λ: also increases like a clock hand → not conserved "integral"

→ The effect of M_j on M_i results in additional accelerations. These additional accelerations can be obtained from the gradient of the perturbing potential. This perturbing potential is called the disturbing function

Basic Idea → Fourier analyze the disturbing function as expressed in orbital elements

⇩

isolate the terms of interest, and assume that the time averaged contributions of others are neglible

↙ ↓

1. secular terms 2. resonant terms

↓

3. short period terms.

Derivation of the disturbing function

$$|\vec{r}_i| = r_i = (x_i^2 + y_i^2 + z_i^2)^{1/2}$$

$$|\vec{r}_j| = r_j = (x_j^2 + y_j^2 + z_j^2)^{1/2}$$

$$|\vec{r}_i - \vec{r}_j| = \left[(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2\right]^{1/2}$$

From Newton's law of universal gravity in the inertial frame we get:

$$m_c \ddot{\vec{R}}_c = G m_c m_i \frac{\vec{r}_i}{r_i^3} + G m_c m_j \frac{\vec{r}_j}{r_j^3}$$

$$m_i \ddot{\vec{R}}_i = G m_i m_j \frac{(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^3} - G m_i m_c \frac{\vec{r}_i}{r_i^3}$$

$$m_j \ddot{\vec{R}}_j = G m_j m_i \frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^3} - G m_j m_c \frac{\vec{r}_j}{r_j^3}$$

Accelerations relative to the primary are given by:

$$\ddot{\vec{r}}_i = \ddot{\vec{R}}_i - \ddot{\vec{R}}_c \qquad \ddot{\vec{r}}_j = \ddot{\vec{R}}_j - \ddot{\vec{R}}_c$$

substituting the expressions for $\ddot{\vec{R}}_c$ $\ddot{\vec{R}}_i$ $\ddot{\vec{R}}_j$

recall little r's are relative to the primary

$$\ddot{\vec{r}}_i = -G(M_c + m_i)\frac{\vec{r}_i}{r_i^3} + Gm_j\left(\frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|^3} - \frac{r_j}{r_j^3}\right)$$

(and similarly for $\ddot{\vec{r}}_j$)

This relative acceleration can be written as gradients of scalar potential functions: (time dependant)

$$\ddot{\vec{r}}_i = \nabla_i(U_i + R_i) = \left(\hat{i}\frac{\partial}{\partial x_i} + \hat{j}\frac{\partial}{\partial y_i} + \hat{k}\frac{\partial}{\partial z_i}\right)(U_i + R_i)$$

* note $\vec{r}_i$ not a function of x_j, y_j, z_j

$$U_i = \frac{G(M_c + m_i)}{r_i} \qquad \boxed{R_i = \frac{Gm_j}{|\vec{r}_j - \vec{r}_i|} - Gm_j\frac{\vec{r}_i \cdot \vec{r}_j}{r_j^3}}$$

Direct Term — Indirect term

The Disturbing Function

The indirect term arises from choice of origin. When the origin is at the center of mass, the indirect term disappears.

→ consider a star–2planet system.

secondaries $= m, m'$

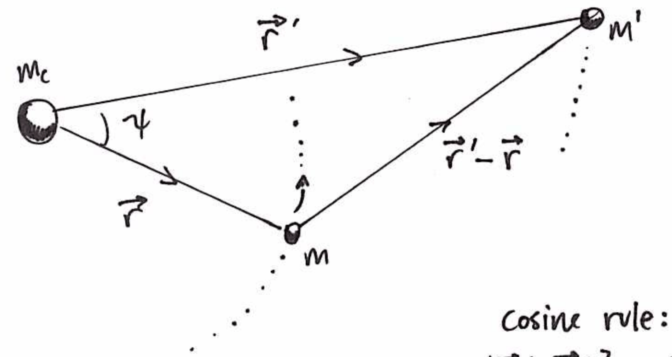
radii $= r, r'$ with $r < r'$ always

EOM for inner secondary is: $\ddot{\vec{r}} + G(M_c + m)\frac{\vec{r}}{r^3} = Gm'\left(\frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} - \frac{\vec{r}'}{r'^3}\right)$

For this 2-body case, the disturbing function is

$$R = \frac{\mu'}{|\vec{r}' - \vec{r}|} - \mu' \frac{\vec{r} \cdot \vec{r}'}{r'^3}$$

→ we want to get an expression for R in terms of orbital elements.



Cosine rule:

$$|\vec{r}' - \vec{r}|^2 = r^2 + r'^2 - 2rr'\cos\psi$$

$$\frac{1}{|\vec{r}' - \vec{r}|} = \frac{1}{r'}\left[1 - \frac{2r}{r'}\cos\psi + \left(\frac{r}{r'}\right)^2\right]^{-\frac{1}{2}}$$

↙

This can be approximated by taking the first terms in a series of Legendre Polynomials

in the n-body codes, this is what derivs is spending all its time computing →

$$\frac{1}{|\vec{r}' - \vec{r}|} = \frac{1}{r'}\sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^{\ell} P_\ell(\cos\psi)$$

$$P_0(\cos\psi) = 1$$

$$P_1(\cos\psi) = \cos\psi$$

$$P_2(\cos\psi) = \frac{1}{2}(3\cos^2\psi - 1)$$

$$\vdots$$

• Because $\vec{r}\cdot\vec{r}' = rr'\cos\psi = rr' P_1(\cos\psi)$

we can get rid of the term $-\mu' \frac{\vec{r}\cdot\vec{r}'}{r'^3}$ in the disturbing function

$$R = \frac{\mu'}{|\vec{r}' - \vec{r}|} - \mu' \frac{\vec{r}\cdot\vec{r}'}{r'^3}$$

by canceling with second term in the Legendre series.

• The $P_0 \cos\psi$ term (first term) in the Legendre series can be omitted because it does not depend on r, and we are ultimately interested in the gradient of R at r.

Can thus write $$R = \frac{\mu'}{r'} \sum_{l=2}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\psi)$$

→ Remember, we want to expand the disturbing function in terms of orbital elements instead of Cartesian coordinates.

Why? Because for small perturbations, the orbit is basically keplerian. λ is the only element that will change very much. a, e, I, ϖ, and Ω will vary slowly in time, meaning that an accurate description need not involve many terms in the expansions.

Recall λ is the mean longitude.

We will show that the expansion of $\mathcal{R}$ has the form:

$$\mathcal{R} = \mu' \sum_{i=1}^{\infty} S_i(a, a', e, e', I, I') \cos \phi_i$$

where the ϕ_i's are linear combinations of the angle-based orbital elements $\lambda, \lambda', \Omega, \Omega', \varpi, \varpi'$ with the general form:

$$\phi_i = j_{1_i}\lambda' + j_{2_i}\lambda + j_{3_i}\varpi' + j_{4_i}\varpi + j_{5_i}\Omega' + j_{6_i}\Omega$$

where the j_{m_i} $(m = 1, 2, \ldots, 6)$ are integers and

$$\sum_{M=1}^{6} j_{m_i} = 0$$

To see how this expansion works, consider two planets in the same orbital plane $\rightarrow I = 0, \Omega = 0$

write ψ as the difference in true longitudes:

$$\psi = (f' + \varpi') - (f + \varpi)$$

f' and f = true anomalies

From trig identities

$$\cos \psi = (\cos f' \cos \varpi' - \sin f' \sin \varpi)(\cos f \cos \varpi - \sin f \sin \varpi)$$
$$+ (\sin f' \cos \varpi' + \cos f' \sin \varpi')(\sin f \cos \varpi + \cos f \sin \varpi)$$

sin f and cos f can be written in terms of series expansions;

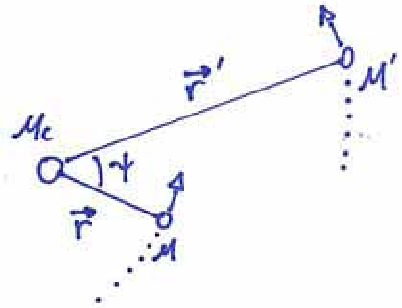
$$\sin f = \sin M + e\sin 2M + e^2\left(\frac{9}{8}\sin 3M - \frac{7}{8}\sin M\right)$$
$+ \ldots$ 3rd order and higher in e

$$\cos f = \cos M + e(\cos 2M - 1) + \frac{9e^2}{8}(\cos 3M - \cos M)$$
$+ \ldots$ 3rd order and higher in e

Using these, get, to 2nd order in e:

$$\begin{aligned}
\cos\psi = {} & \left(1 - e^2 - e'^2\right)\cos[M - M' + \varpi - \varpi'] \\
& - e\cos[M' - \varpi + \varpi'] - e'\cos[M + \varpi - \varpi'] \\
& + e\cos[2M - M' + \varpi - \varpi'] + e'\cos[M - 2M' + \varpi - \varpi'] \\
& - \frac{1}{8}e^2\cos[M + M' - \varpi + \varpi'] - \frac{1}{8}e'^2\cos[M + M' + \varpi - \varpi'] \\
& + \frac{9}{8}e^2\cos[3M - M' + \varpi - \varpi'] + \frac{9}{8}e'^2\cos[M - 3M' + \varpi - \varpi'] \\
& + ee'\cos[\varpi - \varpi'] + ee'\cos[2M - 2M' + \varpi - \varpi'] \\
& - ee'\cos[2M + \varpi - \varpi'] - ee'\cos[2M' - \varpi + \varpi'].
\end{aligned}$$

How do resonances emerge from the disturbing function?



we showed
that the acceleration of body μ:

$$\ddot{\vec{r}} = \nabla(U + R)$$

U ← potential from central object

R ← disturbing function

$$R = \frac{\mu'}{|\vec{r}' - \vec{r}|} - \frac{\mu' \, \vec{r} \cdot \vec{r}'}{r'^3}$$

express $\frac{1}{|\vec{r}' - \vec{r}|}$ in terms of Legendre Polynomials:

$$R = \frac{\mu'}{r'} \sum_{l=2}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\psi)$$

note

$$\lambda = M + \varpi$$

λ: mean longitude

M: mean anomaly

ϖ: longitude of periastron

Ω = longitude of ascending node ⇨ longitudes are measured wrt reference line.

This series expansion for the disturbing function can be written

$$R = \mu' \sum S(a, a', e, e', I, I') \cos \phi$$

where the general form for the argument φ is

$$\varphi = (\ell - 2p' + q')\lambda' - (\ell - 2p + q)\lambda - q'\varpi' + q\varpi + (m - \ell + 2p')\Omega' - (m - \ell + 2p)\Omega$$

where ℓ, m, p, p', q, q' are all integers

rewriting $\quad \varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$

it must be true that

$$\sum_{i=1}^{6} j_i = 0$$

(provided that longitudes are used.) ↗ not anomalies

note that a, a', e, e', I, I' should all be slowly varying. Hence, in general, if an argument φ in the series contains λ or λ', then it cycles rapidly through $0 \to 2\pi$, and the contributions to R average out.

Next, turn attention to the strengths, S, of the individual terms

$$R = \frac{\mu'}{r'} \sum_{\ell=2}^{\infty} \left(\frac{r}{r'}\right)^{\ell} P_\ell(\cos\psi) = \frac{\mu'}{a'} \sum_{\ell=2}^{\infty} \alpha^{\ell} \left(\frac{a'}{r'}\right)^{\ell+1} \left(\frac{r}{a}\right)^{\ell} P_\ell(\cos\psi)$$

where $\alpha = a/a'$ = ratio of the semi-major axes.

In order to completely express $\mathcal{R}$ in terms of orbital elements we need to express the distance r in terms of the orbital elements

→ recall $a(1+e) = r_{apastron}$

$a(1-e) = r_{periastron}$

$$\frac{r}{a} = 1 - e\cos M + \frac{e^2}{2}(1 - \cos 2M) + \frac{3e^3}{8}(\cos M - \cos 3M) + \cdots$$

substitute this series into Legendre Polynomial form for $\mathcal{R}$, and turn the algebraic crank.

$$\begin{aligned}\mathcal{R} = \frac{\mu'}{a'}\sum_{l=2}^{\infty}\alpha^l \sum_{m=0}^{l}(-1)^{l-m}\kappa_m \frac{(l-m)!}{(l+m)!} \\ \times \sum_{p,p'=0}^{l} F_{lmp}(I)F_{lmp'}(I') \sum_{q,q'=-\infty}^{\infty} X_{l-2p+q}^{l,l-2p}(e) X_{l-2p'+q'}^{-(l+1),l-2p'}(e') \\ \times \cos\bigl[(l-2p'+q')\lambda' - (l-2p+q)\lambda - q'\varpi' + q\varpi \\ + (m-l+2p')\Omega' - (m-l+2p)\Omega\bigr],\end{aligned} \quad (6.36)$$

Everything in this expression **is written in terms of orbital elements!**

where $\alpha = a/a'$, λ and λ' are mean longitudes, ϖ and ϖ' are the longitudes of pericentre, and $\kappa_0 = 1$ and $\kappa_m = 2$ for $m \neq 0$.

The $F_{lmp}(I)$ are the inclination functions defined as

$$F_{lmp}(I) = \frac{\mathrm{i}^{l-m}(l+m)!}{2^l p!(l-p)!} \times \sum_k (-1)^k \binom{2l-2p}{k}\binom{2p}{l-m-k} c^{3l-m-2p-2k} s^{m-l+2p+2k},$$

where $\mathrm{i} = \sqrt{-1}$, k is summed from $k = \max(0, l-m-2p)$ to $k = \min(l-m, 2l-2p)$, $s = \sin\frac{1}{2}I$, and $c = \cos\frac{1}{2}I$.

The quantities $X_c^{a,b}(e)$ are *Hansen coefficients*, which can be defined by

$$X_c^{a,b}(e) = e^{|c-b|}\sum_{\sigma=0}^{\infty} X_{\sigma+\alpha,\sigma+\beta}^{a,b} e^{2\sigma}.$$

In this context $\alpha = \max(0, c-b)$, $\beta = \max(0, b-c)$, and the $X_{c,d}^{a,b}$ are *Newcomb operators*, which can be defined recursively by

$$X_{0,0}^{a,b} = 1,$$
$$X_{1,0}^{a,b} = b - a/2,$$

continued on next page.

and, for $d = 0$,

$$4cX_{c,0}^{a,b} = 2(2b - a)X_{c-1,0}^{a,b+1} + (b - a)X_{c-2,0}^{a,b+2},$$

or, for $d \neq 0$,

$$\begin{aligned} 4dX_{c,d}^{a,b} = & -2(2b + a)X_{c,d-1}^{a,b-1} - (b + a)X_{c,d-2}^{a,b-2} \\ & - (c - 5d + 4 + 4b + a)X_{c-1,d-1}^{a,b} \\ & + 2(c - d + b)\sum_{j \geq 2}(-1)^j \binom{3/2}{j} X_{c-j,d-j}^{a,b}. \end{aligned}$$

Also, $X_{c,d}^{a,b} = 0$ if $c < 0$ or $d < 0$. If $d > c$ then $X_{c,d}^{a,b} = X_{d,c}^{a,-b}$.

Arguments that do not involve a mean longitude are the secular terms $\omega - \varpi'$, etc.

Secular terms

However, not all arguments with a mean longitude are necessarily *noncontributing*

write $\lambda = nt + \epsilon$ ← mean longitude at epoch

consider a general argument

$$\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$$

$$j_1\lambda' + j_2\lambda \simeq (j_1 n' + j_2 n)t + \text{constant}$$

So if the semi-major axes are such that

$$j_1 n' + j_2 n = 0$$

→ Then the argument is slowly varying.

Resonant terms

→ Given the form of the disturbing function that we choose, we still need to take its gradient to accelerate the perturbed body. This process can be expressed as continually changing the osculating orbital →

elements of the body through Lagrange's Planetary Equations

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \epsilon}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\left(1-\sqrt{1-e^2}\right)\frac{\partial R}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varpi}$$

$$\frac{d\epsilon}{dt} = \frac{-2}{na}\frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}\left(1-\sqrt{1-e^2}\right)}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial R}{\partial I}$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$

$$\frac{dI}{dt} = \frac{-\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \epsilon} + \frac{\partial R}{\partial \varpi}\right) - \frac{1}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial R}{\partial \Omega}$$

If the arguments are contributing for a long period of time, then there is an opportunity to build up large changes in the orbital elements.

Heueristically, what is it that the resonant terms describe?

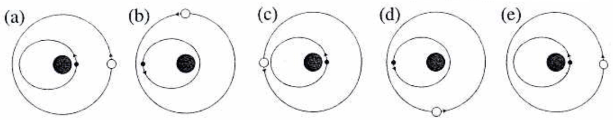


Fig. 8.1. The relative positions of Jupiter (white circle) and an asteroid (small filled circle) for the stable configuration when their orbital periods are in a ratio of 2:1. If T_J is the period of Jupiter's orbit then the diagrams illustrate the configurations at times (a) $t = 0$, (b) $t = \frac{1}{4}T_J$, (c) $t = \frac{1}{2}T_J$, (d) $t = \frac{3}{4}T_J$, and (e) $t = T_J$.

Stable resonant configuration

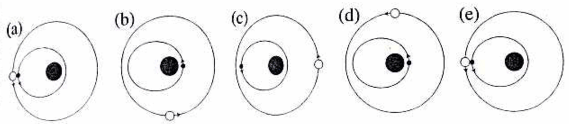
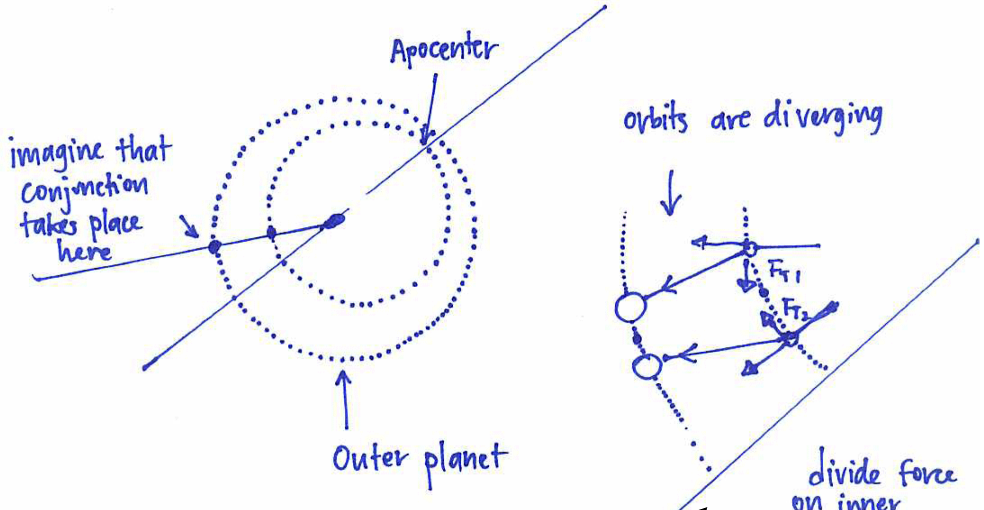


Fig. 8.2. The relative positions of Jupiter (white circle) and an asteroid (small filled circle) for the unstable configuration when their orbital periods are in a ratio of 2:1. If T_J is the period of Jupiter's orbit then the diagrams illustrate the configurations at times (a) $t = 0$, (b) $t = \frac{1}{4}T_J$, (c) $t = \frac{1}{2}T_J$, (d) $t = \frac{3}{4}T_J$, and (e) $t = T_J$.

Unstable resonant configuration

Consider a simple resonant configuration



- F_{t_1} is larger than F_{t_2} and acts for a longer time
- Net effect of encounter increases the energy of the inner particle. → increases period → conjunction occurs later

pericenter

divide force on inner planet into radial, tang. parts.

stabilizing.

Resonant conditions:

general condition $\frac{n'}{n} = \frac{p}{p+q}$ where p and q are integers

if the two bodies are in conjunction at $t=0$, the next conjunction will occur at

$$nt - n't = 2\pi$$

period between conjunctions is

$$T_{con} = \frac{2\pi}{n-n'}$$

but $p(n-n') = q\,n'$

$$T_{con} = \frac{p}{q}\frac{2\pi}{n'} = \frac{p}{q}T' = \frac{p+q}{q}T$$

where T', T are orbital periods of the satellites.

$$q\,T_{con} = p\,T' = (p+q)\,T$$

if $q=1$ then each satellite completes a whole number of orbits between succesive conjunctions, and every conjunction occurs at the same longitude in inertial space

if $q=2$, every other conjunction occurs at the same longitude, etc.

Now consider the possibility that the elliptical orbit is precessing → This screws up the period – conjunction reln.

The resonant relation in the case where precession occurs (for outer orbit only) is

$$(p+q)\,n' - pn - q\dot{\varpi}' = 0$$

so that

$$\frac{n' - \dot{\varpi}'}{n - \dot{\varpi}'} = \frac{p}{p+q}$$

↳ in a frame rotating with the pericenter of the outer satellite, conjunctions are occuring at the same longitude

· In general, both orbits are precessing

For 2:1 mean motion resonance, the arguments are:

$$\varphi_1 = 2\lambda' - \lambda - \varpi'$$

$$\varphi_2 = 2\lambda' - \lambda - \varpi$$

For 3:1 mean motion resonance, the arguments are:

$$\varphi_1 = 3\lambda' - \lambda - 2\varpi'$$

Check these for 55 cancri → $\varphi_2 = 3\lambda' - \lambda - \varpi' - \varpi$

$$\varphi_3 = 3\lambda' - \lambda - 2\varpi$$

If we ignore mutual inclination between the planets, then a mean-motion resonance is present if the "resonant argument" is librating (as opposed to circulating). For co-planar, 2:1 mean motion resonances, these arguments are:

$$\theta_1 = 2\lambda' - \lambda - \varpi$$

$$\theta_2 = 2\lambda' - \lambda - \varpi'$$

↑ note that the sum of the coefficients is zero, as required for a term to contribute to the disturbing function.

EXAMPLE: GJ 876

↓ Three body integration

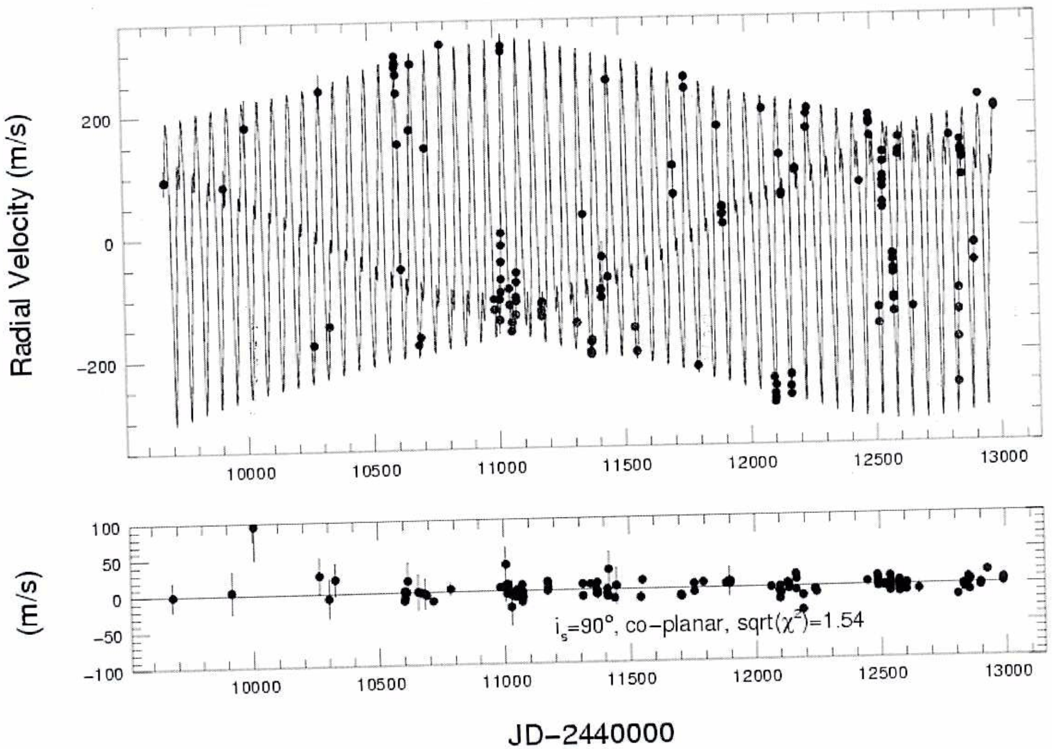
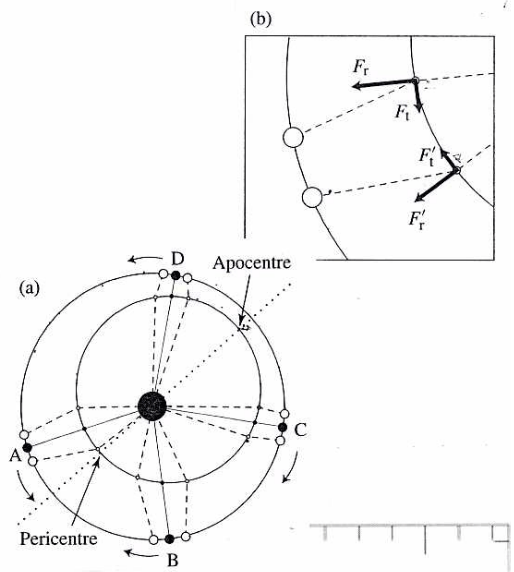
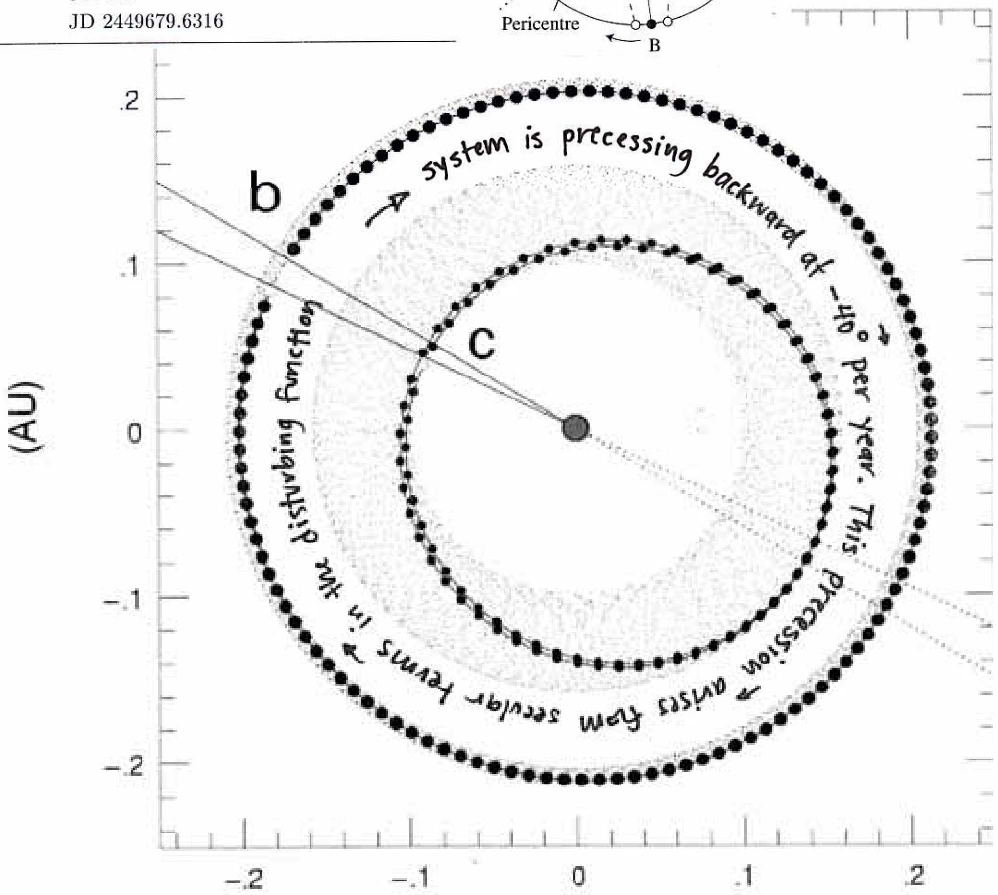


Table 2. Co-Planar Fit to GJ 876 Radial Velocity Data

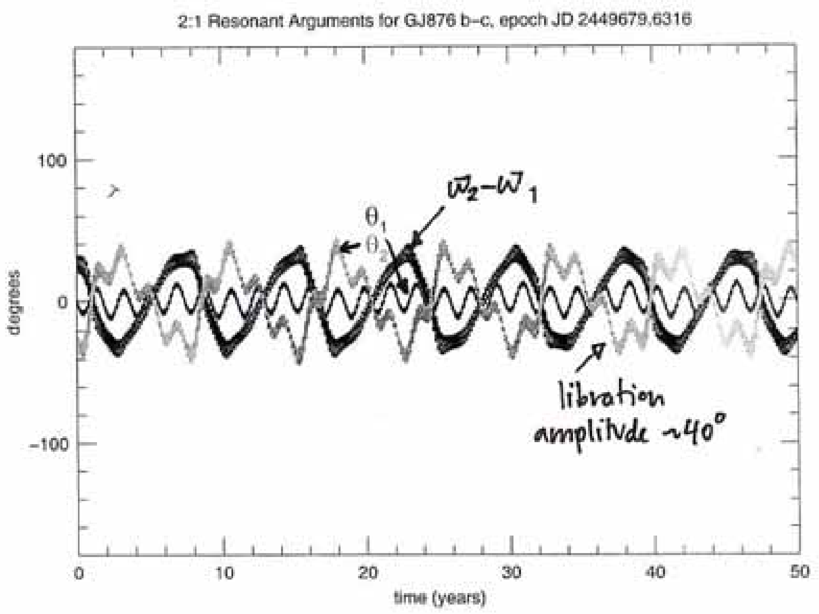
Parameter	Planet c	Planet b
P (d)	30.38 ± 0.03	60.93 ± 0.03
M	0 ± 15°	186 ± 13°
e	0.218 ± 0.002	0.029 ± 0.005
i fixed	90.0°	90.0°
ϖ	154.4 ± 2.9°	149.1 ± 13.4°
m	0.597 ± 0.008 M_{Jup}	1.90 ± 0.01 M_{Jup}
o_1 m s^{-1}	-8.732	
o_2 m s^{-1}	44.476	
transit epoch	JD 2453000.57 ± 0.22	
$\lvert\varpi_c - \varpi_b\rvert_{max}$	34 ± 11°	
θ_{1max}	7.0 ± 1.8°	
θ_{2max}	34 ± 12°	
epoch	JD 2449679.6316	





QuickTime™ and a
YUV420 codec decompressor
are needed to see this picture.

GJ 876 integrated for 100 years
(this animation is on the website)



if both θ_1 and θ_2 are librating, then

$\varpi_2 - \varpi_1$ must also librate.

For 3:1 resonance, the co-planar ($\Omega' = \Omega = 0°$) resonant arguments are

$$\theta_1 = 3\lambda' - \lambda' - 2\varpi'$$

$$\theta_2 = 3\lambda' - \lambda' - \varpi' - \varpi$$

$$\theta_3 = 3\lambda' - \lambda' - 2\varpi$$

55 cancri systems.

plot these for the
- JD 2451000
- JD 2452000
- JD 2453000

QuickTime™ and a
YUV420 codec decompressor
are needed to see this picture.

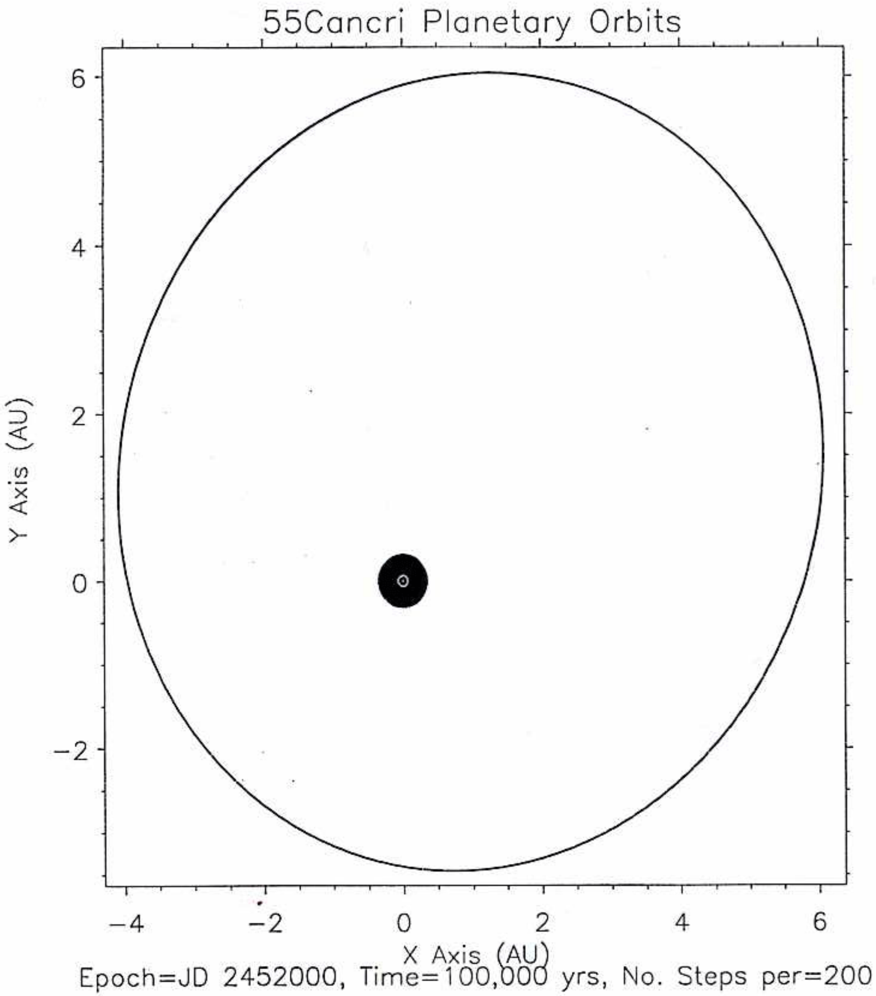
GJ 876 integrated for 100 yea

Music of the spheres?

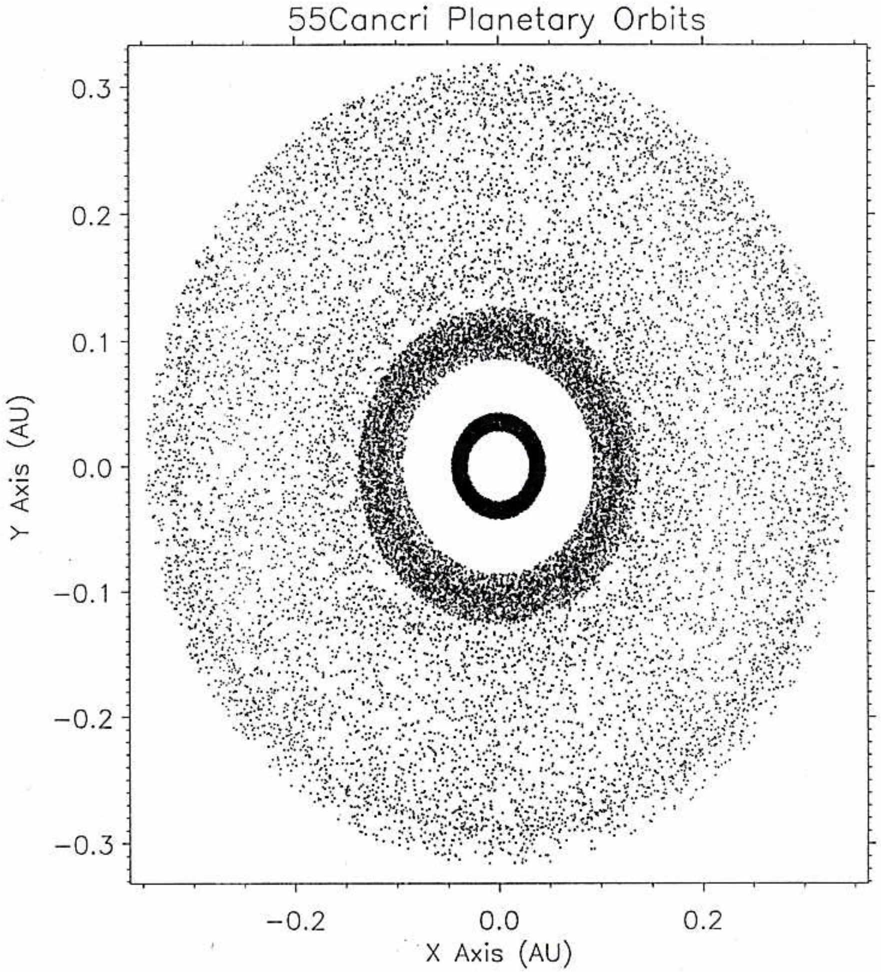
It sounds terrible!

gj876

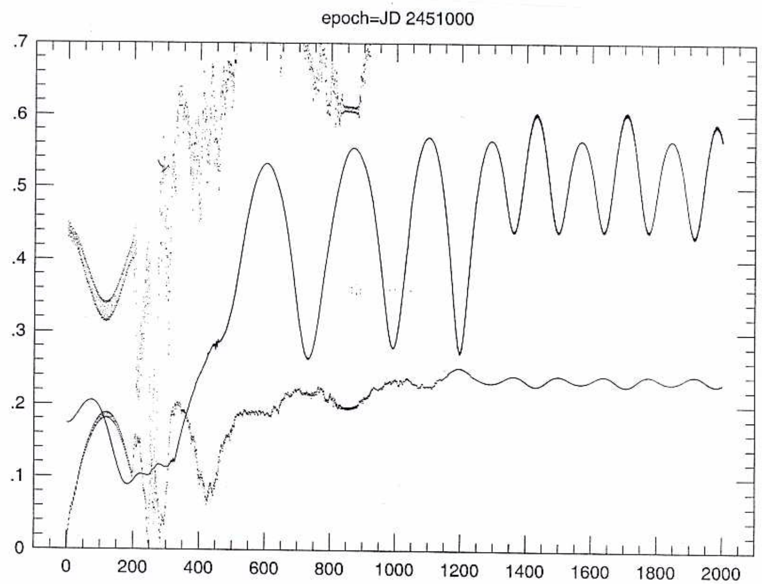
Ups And



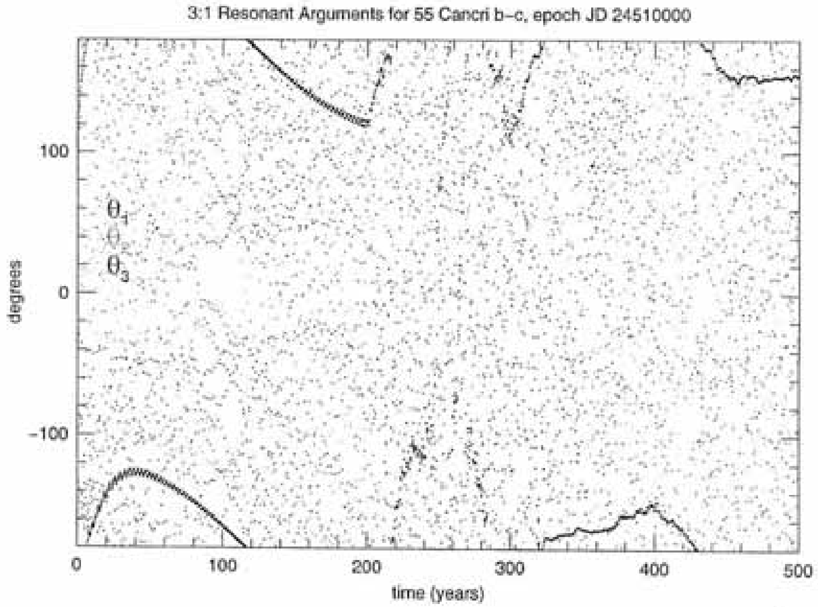
55Cancri Planetary Orbits
Y Axis (AU)
X Axis (AU)
Epoch=JD 2452000, Time=100,000 yrs, No. Steps per=200



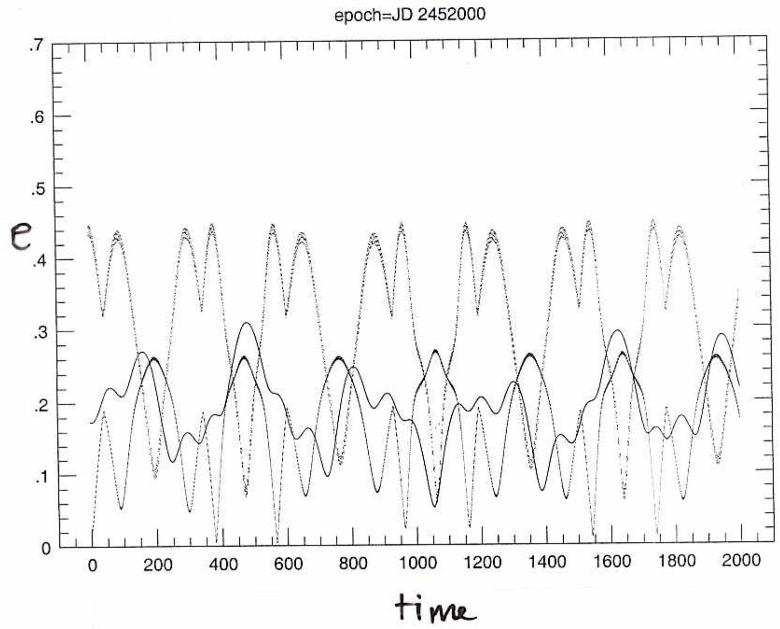
55Cancri Planetary Orbits
Y Axis (AU)
X Axis (AU)



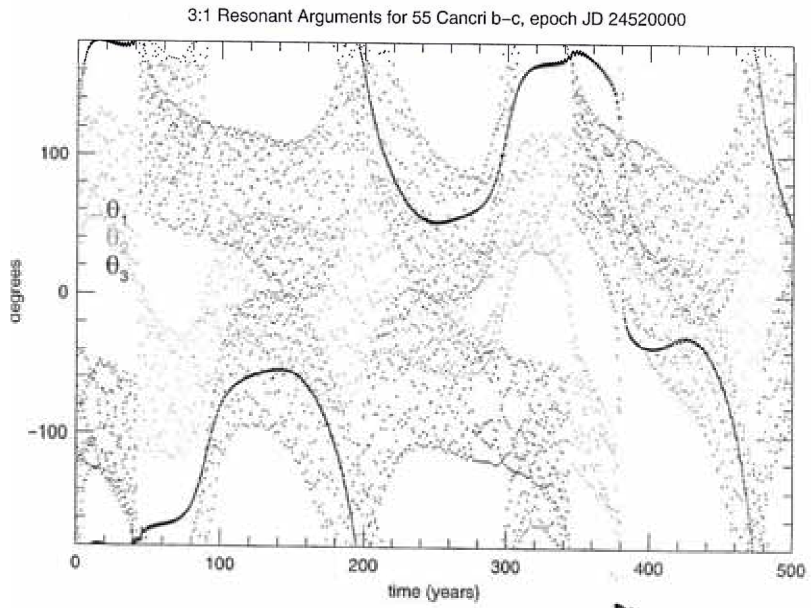
Unstable case where T_{start} = JD 2451000



The 3:1 arguments are all circulating

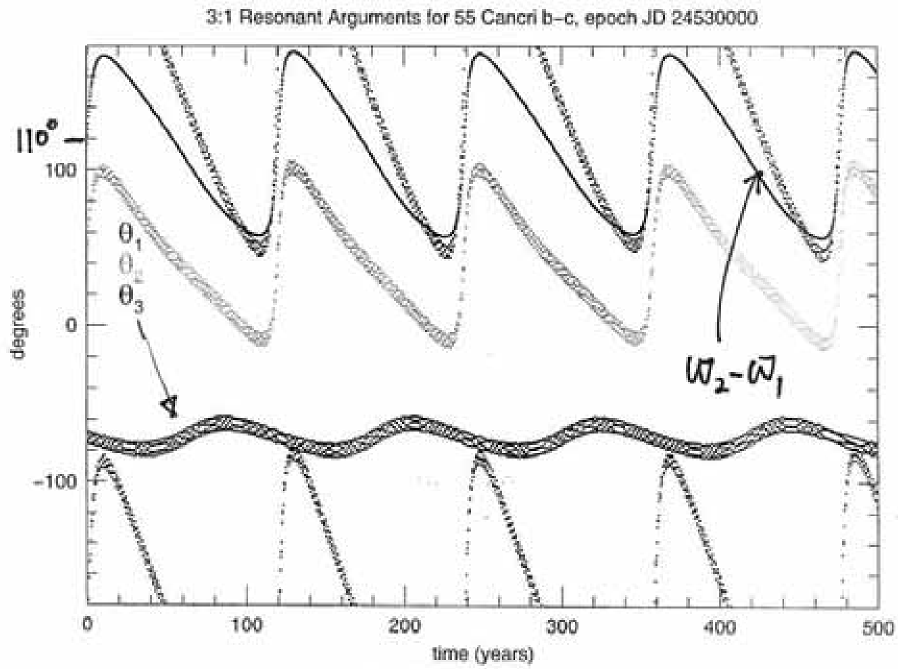


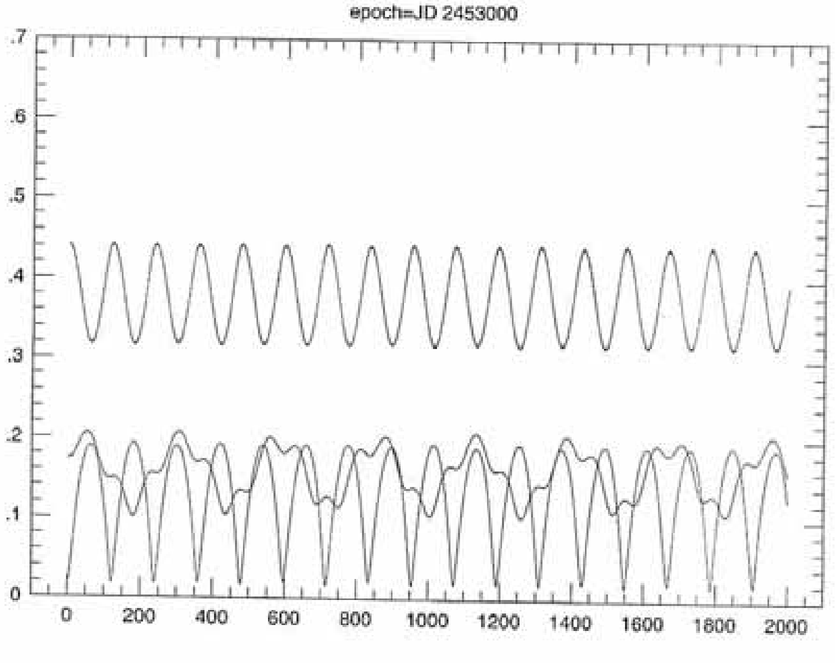
For epoch 2452000 the motion is more complicated



The system is "feeling" the 3:1 resonance, but none of the resonant arguments is truly librating when plotted over 500y

↑ note shorter timescale here than in plot above.





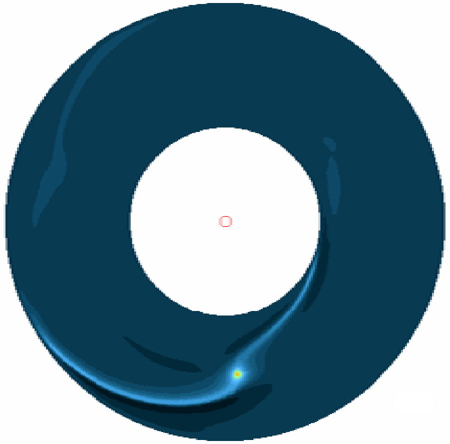
Stable case

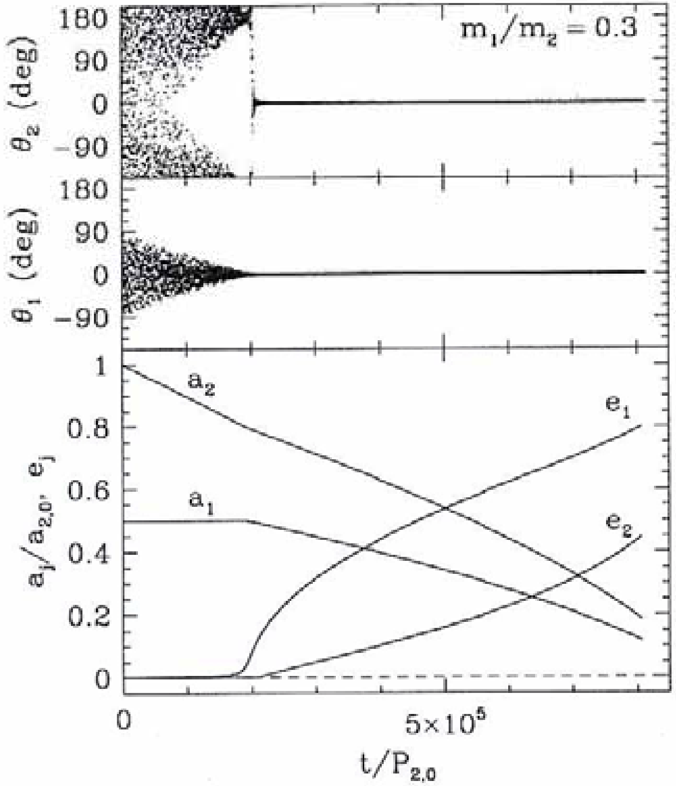
↓

All three arguments are librating!

How do systems get into resonance?

→ Differential Migration → 1. Tidal Evolution
2. Disk Migration

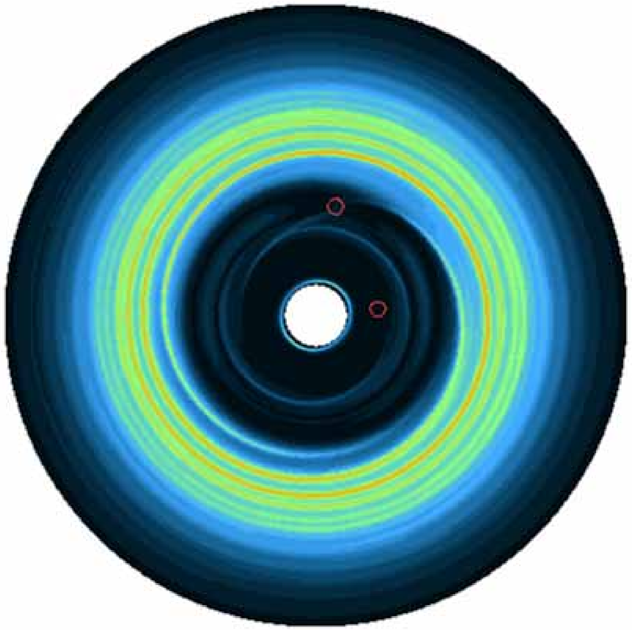




Lee & Peale

↑

Simulation of differential migration for GJ 876



Differential Migration leads to different resonant outcomes, depending on mass ratio and migration rate.

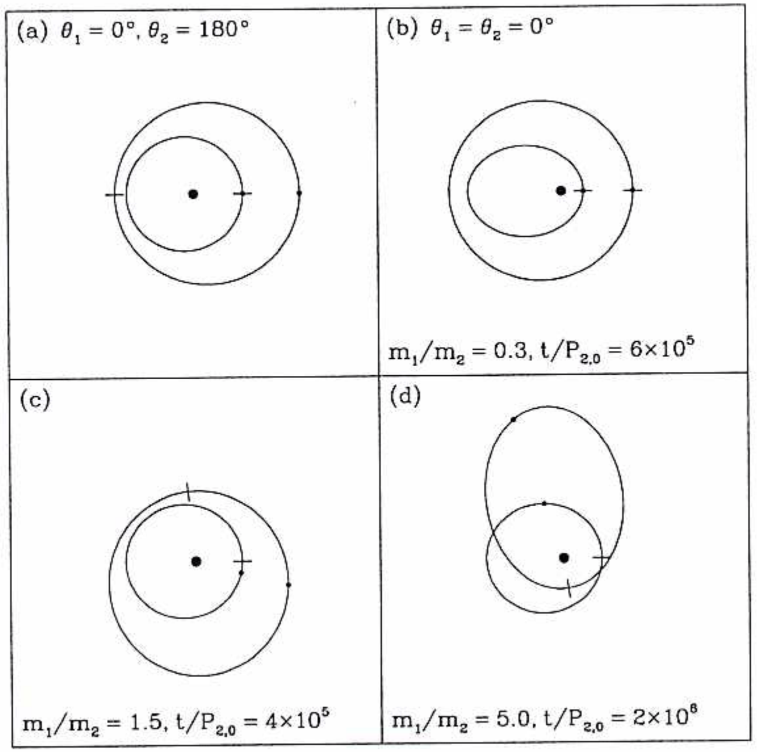
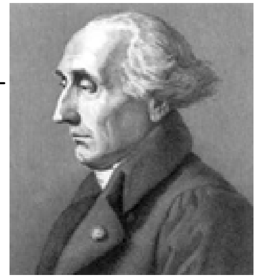


Fig. 2.— Examples of 2:1 resonance configurations that can be reached by differential migration of planets with constant masses and initially nearly circular orbits. The dashes on the ellipses representing the orbits mark the positions of the periapses, and the planets represented by the small dots are shown at conjunction. (*a*) Anti-symmetric configuration with $\theta_1 \approx 0°$ and $\theta_2 \approx 180°$ at small eccentricities. The eccentricities of the orbits are exaggerated so that the positions of the periapses are more visible. (*b*) Symmetric configuration with $\theta_1 \approx \theta_2 \approx 0°$ near $t/P_{2,0} = 6 \times 10^5$ in Fig. 1. (*c*) Asymmetric configuration near $t/P_{2,0} = 4 \times 10^5$ in Fig. 3. (*d*) Asymmetric configuration with intersecting orbits near $t/P_{2,0} = 2 \times 10^6$ in Fig. 4.

QuickTime™ and a
YUV420 codec decompressor
are needed to see this picture.

HD 128311 -- 2:1 resonance. θ_1 librating, θ_2 circulating
(this animation is on the website)

Secular Perturbation Theory



Lagrange Laplace

Understanding Long-Term Interactions

- 1748- Euler develops perturbation framework
  - Describes the effect of planet-planet interactions in terms of time-varying deviations of the orbital elements (i.e. describe the perturbation in terms of a disturbing function.)
- 1770's- Lagrange and Laplace .
  - Divided the disturbing function into *secular* and *periodic* terms. The secular "occuring over an age" terms arise from treating the planetary orbit as a wire of varying thickness. The periodic (including resonant) terms depend on the mean longitudes (the positions of the planets in their orbits). The periodic terms were assumed to average out over many orbits.
  - Showed that to second order in inclination and eccentricity, the inclinations and eccentricities of the planets vary periodically, and that the semi-major axes of the planets remain constant. In particular Jupiter and Saturn participate in a 71,000 year exchange of angular momentum -- the Laplace-Lagrange mode.
  - This secular exchange did not explain the great inequality, however.

The Laplace Lagrange Mode

To second order in eccentricities, the disturbing function's

$$R = \mu' \sum_{1}^{\infty} S(a, a', e, e', I, I') \cos\phi$$

with $\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$

Secular terms can be written

1st & second terms have $\cos\phi = 1$

$$R^{secular} = \frac{1}{8}\left[2\alpha_{12}D + \alpha_{12}^2 D^2\right] b_{\frac{1}{2}}^{(0)} (e_1^2 + e_2^2)$$

$$- \frac{1}{2}\alpha_{12} b_{3/2}^{(1)} (s_1^2 + s_2^2) + \alpha_{12} b_{3/2}^{(1)} s_1 s_2 \cos(\Omega_1 - \Omega_2)$$

$$+ \frac{1}{4}\left[2 - 2\alpha_{12}D - \alpha_{12}^2 D^2\right] b_{\frac{1}{2}}^{(1)} e_1 e_2 \cos(\varpi_1 - \varpi_2)$$

DEFS:

$\alpha_{12} = \frac{a_1}{a_2}$

$D = \frac{d}{d\alpha_{12}}$

$s = \sin\frac{1}{2}I$

$R^{secular}$ has similar form for the two disturbing functions:

$$R_1 = \frac{GM_2}{a_1}\alpha_{12} R^{secular}$$

$$R_2 = \frac{GM_1}{a_2} R^{secular}$$

The $b_{\frac{1}{2}}$, $b_{\frac{3}{2}}$, etc. are Laplace coefficients:

$$\frac{1}{2} b_s^{(j)}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos j\psi \, d\psi}{(1 - 2\alpha\cos\psi + \alpha^2)^{1/2}}$$

With a bunch of algebra, we can write, for R_1 and R_2

$$R_j = n_j a_j^2 \left[\frac{1}{2} A_{jj} e_j^2 + A_{jk} e_1 e_2 \cos(\varpi_1 - \varpi_2) + \frac{1}{2} B_{jj} I_j^2 + B_{jk} I_1 I_2 \cos(\Omega_1 - \Omega_2) \right]$$

where $j = 1,2$; $k = 2,1$ $(j \neq k)$; and

$$A_{jj} = +n_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(1)}(\alpha_{12})$$

$$A_{jk} = -n_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(2)}(\alpha_{12})$$

$$B_{jj} = -n_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(1)}(\alpha_{12})$$

$$B_{jk} = +n_j \frac{1}{4} \frac{m_k}{m_c + m_j} \alpha_{12} \bar{\alpha}_{12} b_{3/2}^{(1)}(\alpha_{12})$$

(where $\bar{\alpha}_{12} = \alpha_{12}$ if $j=1$; $\bar{\alpha}_{12} = 1$ if $j=2$)

All of these quantities are constants which can be arranged into two matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Recall that Laplace's planetary equations tell us how to time-advance the orbital elements given the choice of terms used in the disturbing function

$$\frac{de_j}{dt} = -\frac{1}{n_j a_j^2 e_j}\frac{\partial R_j}{\partial \varpi_j} \; ; \quad \frac{d\varpi_j}{dt} = +\frac{1}{n_j a_j^2 e_j}\frac{\partial R_j}{\partial e_j}$$

$$\frac{dI_j}{dt} = -\frac{1}{n_j a_j^2 I_j}\frac{\partial R_j}{\partial \Omega_j} \; ; \quad \frac{d\Omega_j}{dt} = +\frac{1}{n_j a_j^2 I_j}\frac{\partial R_j}{\partial I_j}$$

To find an analytic solution

define eccentricity & inclination "vectors"

$$h_j = e_j \sin\varpi_j \qquad k_j = e_j \cos\varpi_j$$

$$p_j = I_j \sin\Omega_j \qquad q_j = I_j \cos\Omega_j$$

The general secular part of the disturbing function can then be written

$$R_j = n_j a_j^2 \Big[\tfrac{1}{2} A_{jj}(h_j^2 + k_j^2) + A_{jk}(h_j h_k + k_j k_k) + \tfrac{1}{2} B_{jj}(p_j^2 + q_j^2) + B_{jk}(p_j p_k + q_j q_k) \Big]$$

The planetary equations can be written (applying the chain rule)

$$\frac{dh_j}{dt} = \frac{\partial h_j}{\partial e_j}\frac{de_j}{dt} + \frac{\partial h_j}{\partial \varpi_j}\frac{d\varpi_j}{dt}$$

$$\frac{dk_j}{dt} = \frac{\partial k_j}{\partial e_j}\frac{de_j}{dt} + \frac{\partial k_j}{\partial \varpi_j}\frac{d\varpi_j}{dt}$$

$$\frac{dp_j}{dt} = \frac{\partial p_j}{\partial I_j}\frac{dI_j}{dt} + \frac{\partial p_j}{\partial \Omega_j}\frac{d\Omega_j}{dt}$$

$$\frac{dq_j}{dt} = \frac{\partial q_j}{\partial I_j}\frac{dI_j}{dt} + \frac{\partial q_j}{\partial \Omega_j}\frac{d\Omega_j}{dt}$$

where the partial derivatives are (following from the defns.)

$$\frac{\partial h_j}{\partial e_j} = \frac{h_j}{e_j}, \quad \frac{\partial k_j}{\partial e_j} = \frac{k_j}{e_j}, \quad \frac{\partial h_j}{\partial \varpi_j} = k_j, \quad \frac{\partial k_j}{\partial \varpi_j} = -h_j$$

$$\frac{\partial p_j}{\partial I_j} = \frac{p_j}{I_j}, \quad \frac{\partial q_j}{\partial I_j} = \frac{q_j}{I_j}, \quad \frac{\partial p_j}{\partial \Omega_j} = q_j, \quad \frac{\partial q_j}{\partial \Omega_j} = -p_j$$

After more algebraic manipulation, find that the variation in orbital elements can be written

$$\dot{h}_j = \frac{1}{n_j a_j^2}\frac{\partial R_j}{\partial k_j}, \quad \dot{k}_j = \frac{-1}{n_j a_j^2}\frac{\partial R_j}{\partial h_j}$$

$$\dot{p}_j = \frac{1}{n_j a_j^2}\frac{\partial R_j}{\partial q_j}, \quad \dot{q}_j = \frac{-1}{n_j a_j^2}\frac{\partial R_j}{\partial p_j}$$

writing this out explicitly for bodies 1 and 2,

$$\dot{h}_1 = A_{11}k_1 + A_{12}k_2 \qquad \dot{k}_1 = -A_{11}h_1 - A_{12}h_2$$
$$\dot{h}_2 = A_{21}k_1 + A_{22}k_2 \qquad \dot{k}_2 = -A_{21}h_1 - A_{22}h_2$$
$$\dot{p}_1 = B_{11}q_1 + B_{12}q_2 \qquad \dot{q}_1 = -B_{11}p_1 - B_{12}p_2$$
$$\dot{p}_2 = B_{21}q_1 + B_{22}q_2 \qquad \dot{q}_2 = -B_{21}p_1 - B_{22}p_2$$

1. time variation of $h_j, k_j \leftrightarrow e, \varpi$ is decoupled from $p_j, q_j \leftrightarrow I, \Omega$

2. These are linear, first order, ordinary differential equations with constant coefficients. Hence, the problem of secular perturbations reduces to two sets of eigenvalue problems!

The solutions are given by

$$h_j = \sum_{i=1}^{2} e_{ji} \sin(g_i t + \beta_i) \; ; \quad k_j = \sum_{i=1}^{2} e_{ji} \cos(g_i t + \beta_i)$$

$$p_j = \sum_{i=1}^{2} I_{ji} \sin(f_i t + \gamma_i) \; ; \quad q_j = \sum_{i=1}^{2} I_{ji} \cos(f_i t + \gamma_i)$$

The frequencies g_i $(i=1,2)$ are the eigenvalues of the matrix **A** with e_{ji} the components of the two corresponding eigenvectors. The frequencies f_i $(i=1,2)$ are the eigenvalues of the matrix **B**, with I_{ji} the components of the corresponding eigenvectors.

The phases β_i and γ_i as well as the amplitudes of the eigenvectors are determined by the initial conditions. That is, the eccentricities inclinations, nodes and arguments of periastron at time $t=0$

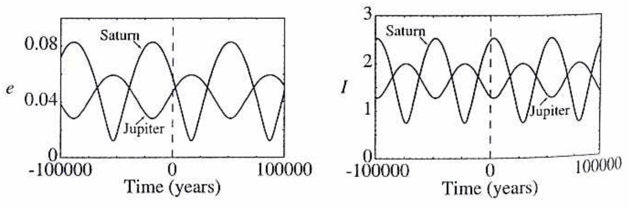
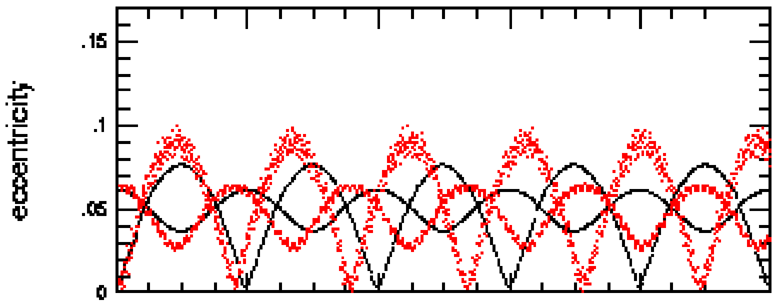


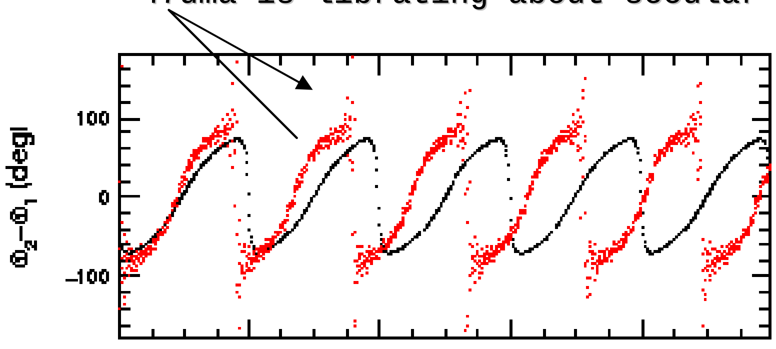
Fig. 7.1. The (a) eccentricities and (b) inclinations of Jupiter and Saturn derived from a secular perturbation theory calculated over a time span of 200,000 y centred on 1983.

The laplace-lagrange theory does a pretty good job of describing Jupiter-Saturn, as well as many of the non-resonant multi-planet exosystems

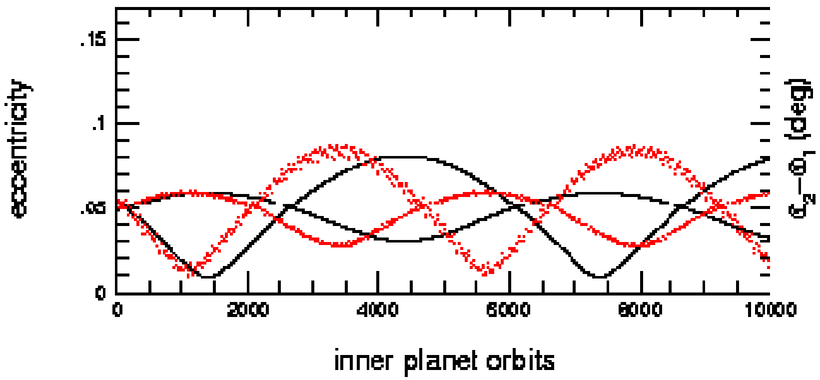
47 Uma “b” and “c”



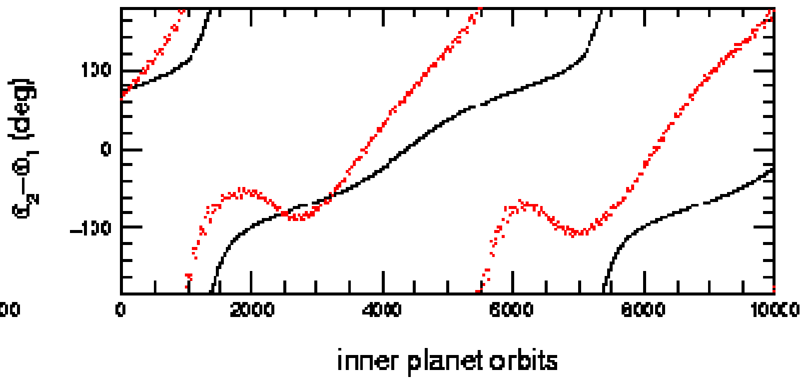
47uma is librating about secular resonance



Jupiter-Saturn



Jupiter and Saturn are circulating



Red = N-body integrations

Black = Laplace-Lagrange secular theory

On the website is a code that computes the laplace-lagrange theory for an arbitrary planetary system.

QuickTime™ and a
YUV420 codec decompressor
are needed to see this picture.

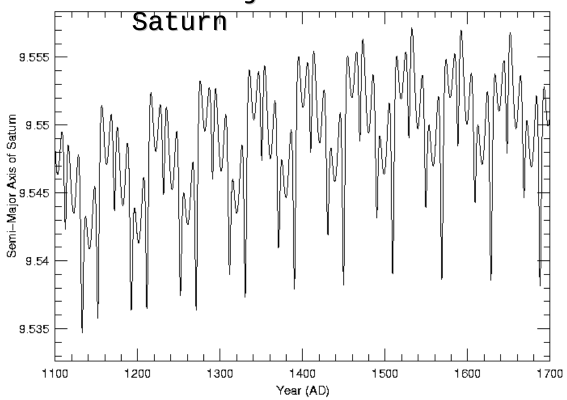
HD 37124

The Great Inequalilty Explained

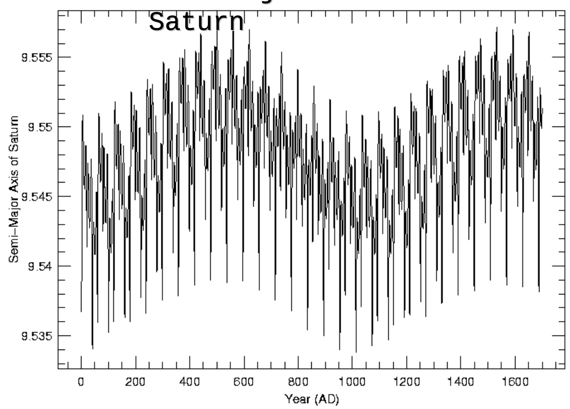
1776: Laplace discovers the source of the Great Inequality as arising from the 5:2 near-resonance between Jupiter and Saturn.

- He showed that a previously neglected third-order term in the periodic portion of the disturbing function has a very small denominator. This term leads to a 926 year periodicity in the mean motions of Jupiter and Saturn. The apparent long-term accelerations of the orbits of Jupiter and Saturn are actually periodic.
- With his theory of the great inequality, Laplace was able to explain 2000 year old Chaldean observations of Saturn. He asserted that the Solar System is indefinitely stable.

Laplace



What appeared to be a steady change in period…



Is actually a ~1000 year periodic variation.

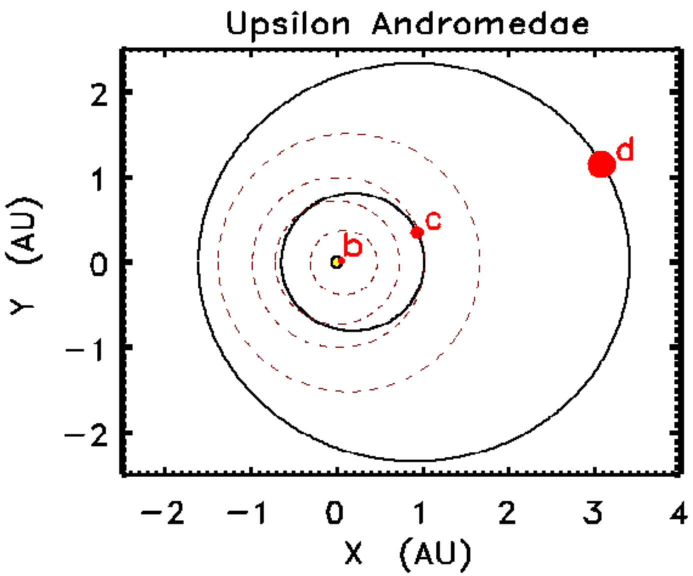
“These laws which thus regulate the eccentricities and inclinations of the planetary orbits, combined with the invariability of the mean distances, secure the permanence of the solar system throughout an indefinite lapse of ages, and offer to us an impressive indication of the Supreme Intelligence”

-Robert Grant “A History of Physical Astonomy”, 1852

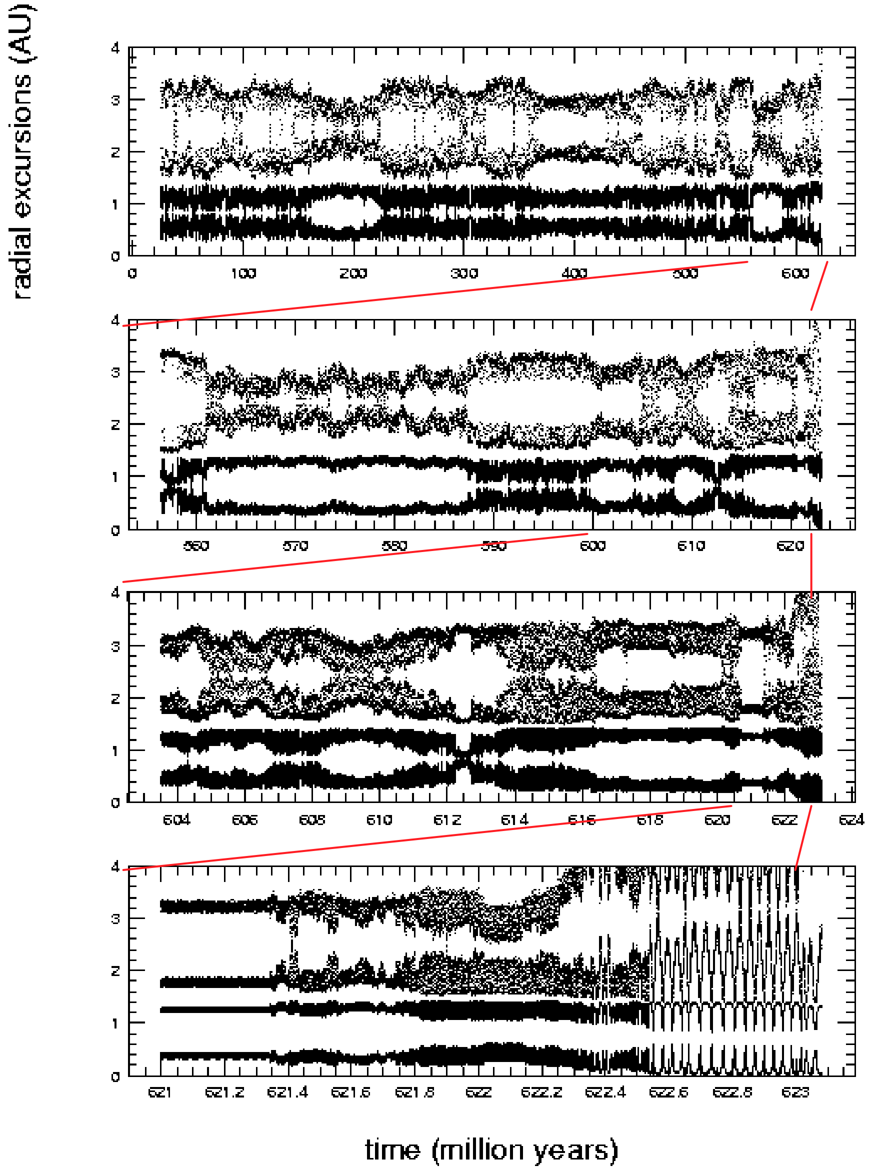
QuickTime™ and a
Video decompressor
are needed to see this picture.

chaos

By the late 1980's it was established that the solar system is weakly chaotic (Sussman & Wisdom 1988, Laskar 1988). Much stronger chaos is often present in observationally allowed configurations of multiple-planet exoplanetary systems. Dynamical integrations can be used to rule out large chunks of observationally allowed parameter space.



Fischer et al. 1999



Chaotic interactions between planets in multiple-planet systems can lead to all sorts of disasters (collisions, ejections, scattering). Chaotic evolution likely plays a role in dynamically sculpting many of the systems that are observed today.

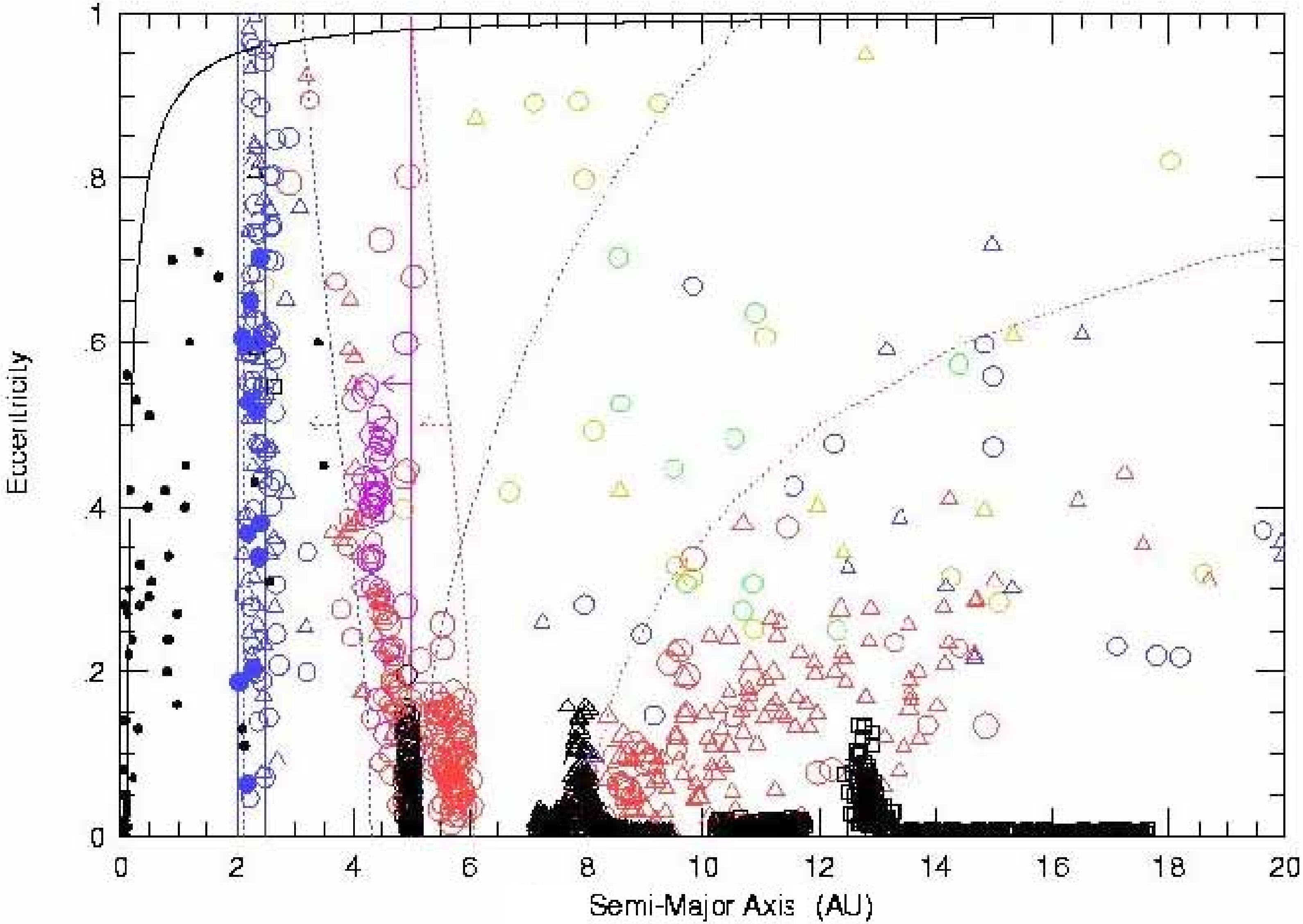


Diagram by Lacy et al (2001), see also Ford et al (2001), Marzari & Weidenschilling (2002), Terquem & Papaloizou (2002), Adams & Laughlin (2003)